

# Singular values of the Dirac operator in dense QCD-like theories

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**ABSTRACT:** We study the singular values of the Dirac operator in dense QCD-like theories at zero temperature. The Dirac singular values are real and nonnegative at any nonzero quark density. The scale of their spectrum is set by the diquark condensate, in contrast to the complex Dirac eigenvalues whose scale is set by the chiral condensate at low density and by the BCS gap at high density. We identify three different low-energy effective theories with diquark sources applicable at low, intermediate, and high density, together with their overlapping domains of validity. We derive a number of exact formulas for the Dirac singular values, including Banks-Casher-type relations for the diquark condensate, Smilga-Stern-type relations for the slope of the singular value density, and Leutwyler-Smilga-type sum rules for the inverse singular values. We construct random matrix theories and determine the form of the microscopic spectral correlation functions of the singular values for all nonzero quark densities. We also derive a rigorous index theorem for non-Hermitian Dirac operators. Our results can in principle be tested in lattice simulations.

**KEYWORDS:** Spontaneous symmetry breaking, chiral Lagrangians, random matrix theory, sum rules, index theorem

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# 1 Introduction

A prominent feature of quantum chromodynamics (QCD) in the vacuum is the dynamical breaking of chiral symmetry through the formation of a chiral condensate  $\langle \bar{\psi}\psi \rangle$ . While an analytical demonstration of this phenomenon from the underlying theory is still lacking, a number of studies have deepened our understanding of its nature. For example, spontaneous chiral symmetry breaking is reflected in the infrared limit of the Dirac eigenvalue spectrum through the Banks-Casher relation [1] and the Smilga-Stern relation [2]. Also, a universal finite-volume domain (called the  $\varepsilon$ -regime) has been discovered [3] in which the theory becomes zero-dimensional and is governed entirely by the global symmetries of the system. It was shown to lead to an infinite number of constraints on the low-lying Dirac eigenvalues, known as the Leutwyler-Smilga spectral sum rules [4].

Thanks to the universality, rigorous results for the spectral properties of the Dirac operator can be derived from a much simpler chiral random matrix theory (RMT) which has the same global symmetry-breaking pattern as the QCD vacuum (see [5, 6] for reviews). This has advanced our knowledge of the low-lying Dirac eigenvalues to the level of microscopic spectral correlation functions, which eventually made it possible to determine the value of the chiral condensate to high precision by first-principle lattice QCD simulations [7], thus numerically verifying that chiral symmetry is spontaneously broken in the QCD vacuum. There are also other QCD-like theories that can be described by RMT. They can be classified by the Dyson index  $\beta$ . If the Dirac operator commutes with an anti-unitary operator  $T$ , then  $\beta = 1$  if  $T^2 = \mathbb{1}$  and  $\beta = 4$  if  $T^2 = -\mathbb{1}$ . Otherwise  $\beta = 2$ . Equivalently,  $\beta = 1, 2$ , or  $4$  if the representation of the gauge group in which the fermions transform is pseudoreal, complex, or real, respectively.<sup>1</sup>

At large quark chemical potential  $\mu$ ,<sup>2</sup> QCD is believed to exhibit another nonperturbative phenomenon: The attractive interaction between quarks near the Fermi surface leads to color superconductivity characterized by a diquark condensate  $\langle \psi\psi \rangle$  (see [9, 10] for recent reviews). In the color-superconducting phases, gluons acquire masses through the Anderson-Higgs mechanism, and weakly interacting quarks acquire a Bardeen-Cooper-Schrieffer (BCS) gap  $\Delta$ . In particular, in the color-flavor-locked (CFL) phase [11] of  $N_f = 3$  QCD at high density, chiral symmetry is spontaneously broken in much the same way as in the QCD vacuum, but the order parameter is the four-quark condensate rather than the conventional chiral condensate. Using effective-theory techniques, it was shown [12] that the distribution of the smallest Dirac eigenvalues in the CFL phase is entirely governed by global symmetries and that the relevant scale of the low-lying spectrum is set by the BCS gap  $\Delta$ .

Unfortunately, at  $\mu \neq 0$  the complex phase of the fermion determinant in the QCD partition function has so far hampered lattice computations of physical quantities such as the BCS gap and the diquark condensate. This is an example of the so-called sign problem,

<sup>1</sup>This can be shown following the considerations in sections 21.2–21.4 of [8].

<sup>2</sup>Throughout most of the paper we will use the terms “chemical potential” and “density” interchangeably, unless the distinction is important. Nonzero density always implies nonzero  $\mu$ . However, the reverse is not always true, e.g., in two-color QCD at small  $\mu$  the density is zero for  $\mu < m_\pi/2$ , where  $m_\pi$  is the pion mass.

which is encountered in many areas of physics. However, several QCD-like theories are devoid of the sign problem even at nonzero  $\mu$  [13] and develop a nonzero BCS gap and a diquark condensate (or pion condensate) signaling superfluidity. Examples include two-color QCD ( $\beta = 1$ ),  $\text{Sp}(2N_c)$  gauge theories with fundamental fermions and an arbitrary number  $N_c$  of colors ( $\beta = 1$ ),  $\text{SU}(N_c)$  gauge theories with adjoint fermions ( $\beta = 4$ ), and  $\text{SO}(N_c)$  gauge theories with fundamental fermions ( $\beta = 4$ ). Another example is three-color QCD at nonzero isospin chemical potential ( $\beta = 2$ ).<sup>3</sup> In spite of the clear differences between these theories and three-color QCD at nonzero quark chemical potential, they share some common features with *bona fide* QCD, e.g., the same mechanism of quark-quark pairing that leads to superfluidity or color superconductivity at nonzero  $\mu$ , and the universality of their phase diagrams [14]. This observation leads us to expect that we can obtain some insights into the properties of realistic QCD matter through a deeper understanding of these QCD-like theories.

Indeed, the aforementioned work on the Dirac eigenvalues in the CFL phase has been successfully generalized to two-color QCD at high density ( $\beta = 1$ ) [15], where the BCS pairing was shown to have sizable consequences on the behavior of the small Dirac eigenvalues, as represented through new Leutwyler-Smilga-type sum rules. (This analysis permits a straightforward extension to theories with  $\beta = 2$  and 4.) Moreover, the chiral random matrix theory that describes the Dirac eigenvalue distribution on the scale characterized by the BCS gap was constructed [16] and solved analytically [17, 18]. These results make it possible in principle to compute the BCS gap from Dirac spectra on the lattice, which would directly verify the BCS-type superfluid phase of QCD-like theories.

Although the BCS gap  $\Delta$  and the diquark condensate  $\langle\psi\psi\rangle$  are closely intertwined at high density, they are *a priori* different objects. While it is unclear to what lower densities the BCS-type pairing with well-defined  $\Delta$  will persist, the diquark condensate is still nonvanishing and breaks global symmetries even at lower densities in those QCD-like theories. Some of the Nambu-Goldstone (NG) modes are diquarks charged under baryon number (with mass  $m_\pi$ ) so that they exhibit Bose-Einstein condensation (BEC) for  $\mu > m_\pi/2$  [14, 19–25].<sup>4</sup> The Bose-Einstein condensate at small  $\mu$  consists of strongly coupled bound diquarks, in contrast to the weakly coupled BCS-type diquarks at large  $\mu$ . Despite this qualitative difference, however, the quantum numbers of the condensate are the same at small and large  $\mu$ , so it is natural to expect that there is no phase transition between the two limits. This phenomenon is called (relativistic) BEC-BCS crossover. It passes a number of nontrivial tests [21, 22, 26], and its possible realization in dense quark matter in QCD has been investigated in various model calculations [27–42] (see also [43–48] for related models). In view of the known relation between  $\Delta$  and the Dirac eigenvalues [12, 15], a natural question arises: How is the diquark condensate  $\langle\psi\psi\rangle$  reflected in the

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<sup>3</sup>To evade the sign problem, one has to assume an even number of flavors with degenerate quark masses in the  $\beta = 1$  and  $\beta = 2$  cases. On the other hand, we can take an arbitrary number of flavors with nondegenerate quark masses in the  $\beta = 4$  cases.

<sup>4</sup>In QCD at nonzero isospin chemical potential  $\mu_I$ , the usual pions, which do not have baryon charge but isospin charge, show BEC, i.e.,  $\langle\bar{d}\gamma_5 u\rangle \neq 0$ , for  $\mu_I > m_\pi$ . In this case, however, it is still possible to view this as a BEC of “diquarks” if we switch  $d \rightarrow d' \equiv C\bar{d}^T$ .

Dirac spectrum of QCD at nonzero density?

In this paper, we point out that it is the spectrum of the *singular values* of the Dirac operator  $D(\mu)$ , i.e., the square roots of the eigenvalues of  $D(\mu)^\dagger D(\mu)$ , that carries the information on the diquark condensate  $\langle\psi\psi\rangle$  at any nonzero chemical potential in all QCD-like theories we consider here. By inserting a diquark source (instead of a quark mass)<sup>5</sup> we derive a number of rigorous relationships covering the entire BEC-BCS crossover region, such as Banks-Casher-type relations, Smilga-Stern-type relations, and Leutwyler-Smilga-type spectral sum rules for the Dirac singular values. This significantly extends previous related work [49] in many directions. We also construct chiral random matrix theories and determine the form of the microscopic spectral correlation functions for singular values at all nonzero densities. In most of this paper we will work in the chiral limit.

Note that, although the Dirac singular values and the Dirac eigenvalues coincide at  $\mu = 0$ , they are essentially different objects at nonzero  $\mu$ .<sup>6</sup> First, while the Dirac eigenvalues are complex, the Dirac singular values are always real and nonnegative. Second, while the scale of the small Dirac eigenvalues is governed by the chiral condensate at small  $\mu$  and by the BCS gap at large  $\mu$ , the scale of the small Dirac singular values is governed by the diquark condensate at any  $\mu$ . From the viewpoint of symmetries, the diquark source breaks  $U(1)_B$  (baryon number), and therefore the spectrum of the Dirac singular values characterizes the  $U(1)_B$  symmetry breaking. This is in contrast to the spectrum of the Dirac eigenvalues, which is unrelated to  $U(1)_B$  since neither the quark mass (the source for the chiral condensate at low density) nor the quark mass squared (the source for  $\Delta^2$  at high density) breaks  $U(1)_B$ .

Our results for the Dirac singular values will make it possible to measure the magnitude of the diquark condensate with high precision by lattice QCD simulations at any density. Together with earlier results for the Dirac eigenvalues they lead to a more detailed understanding of superfluidity and the BEC-BCS crossover in QCD-like theories, and hopefully also of the physics of color superconductivity in three-color QCD.

The structure of this paper is as follows. In section 2, we introduce the microscopic theories considered in this paper and write down the general expressions for the partition functions. In section 3, we review basic properties of the eigenvalues and singular values of the Dirac operator and clarify their relation at nonzero chiral chemical potential. Special emphasis is given to the zero modes. In section 4, we derive Banks-Casher-type relations for two-color QCD ( $\beta = 1$ ), QCD at nonzero isospin chemical potential ( $\beta = 2$ ), and QCD with adjoint fermions ( $\beta = 4$ ). In sections 5, 6, and 7, we concentrate on two-color QCD ( $\beta = 1$ ) for simplicity and brevity. The results of these sections admit straightforward generalization to the other QCD-like theories with  $\beta = 2$  and 4. In section 5 we introduce three different low-energy effective Lagrangians with diquark sources applicable at low, intermediate, and high density. In sections 6 and 7, we derive Smilga-Stern-type relations and

<sup>5</sup>The diquark source can also be called Majorana mass while the quark mass is called Dirac mass.

<sup>6</sup>As an aside we mention that at  $\mu = 0$  the spectrum of the so-called Hermitian Wilson-Dirac operator in lattice QCD with Wilson fermions is nothing but the singular value spectrum of the Wilson-Dirac operator owing to the  $\gamma_5$ -Hermiticity of the operator. In this case, the density of the singular values near the origin is proportional to the parity-breaking condensate [50].

Leutwyler-Smilga-type sum rules, respectively, using the effective theories from section 5. In section 7 we also construct chiral random matrix theories that describe the spectrum of the Dirac singular values in the  $\varepsilon$ -regime and determine the form of the microscopic spectral correlation functions. Section 8 contains the conclusions and an outlook.

In appendix A, we summarize our definitions and conventions. Appendix B describes the derivations of the singular value representations of the partition functions for the theories with  $\beta = 1, 2$ , and 4. In appendix C, we comment on the importance of the positivity of the measure, which could be spoiled by the diquark sources. We also discuss QCD inequalities and derive constraints on the symmetry-breaking pattern for positive definite measure. In appendix D, we present careful derivations of the anomaly equation and the extension of the index theorem to  $\mu \neq 0$ , paying special attention to the non-Hermitian nature of the Dirac operator. In appendix E the derivation of eq. (6.19) in the main text is outlined. In appendix F we comment on the random matrix theory for QCD at nonzero isospin chemical potential ( $\beta = 2$ ). Finally, appendix G is devoted to the derivation of eq. (7.52) in the main text.

## 2 Microscopic theories

In this section, we introduce QCD-like theories with the Dyson indices  $\beta = 1, 2$ , and 4, emphasizing the anti-unitary symmetries of the Dirac operator and the global symmetries of the theories. Since the analysis of all three cases is similar, we first examine the  $\beta = 1$  case in detail and then discuss  $\beta = 2$  and  $\beta = 4$  briefly without redundancy. We take two-color QCD and QCD with adjoint fermions as examples for the  $\beta = 1$  and  $\beta = 4$  cases, respectively. The same arguments are readily applicable to  $\text{Sp}(2N_c)$  gauge theory ( $\beta = 1$ ) and  $\text{SO}(N_c)$  gauge theory ( $\beta = 4$ ).

Unless stated otherwise we always work in Euclidean space and at zero temperature. Our definitions and conventions are summarized in appendix A.

### 2.1 Two-color QCD ( $\beta = 1$ )

The Dirac operator of two-color QCD in the presence of a chemical potential is given by

$$D(\mu) = \gamma_\nu D_\nu + \mu \gamma_4 \quad \text{with} \quad D_\nu = \partial_\nu + i A_\nu, \quad (2.1)$$

where  $A_\nu = A_\nu^a \tau_a / 2$  is the gauge field and the  $\tau_a$  are the generators of  $\text{SU}(2)$  color, i.e., the Pauli matrices. For  $\mu = 0$  the Dirac operator is anti-Hermitian, but for  $\mu \neq 0$  it no longer has definite Hermiticity properties since  $\mu \gamma_4$  is Hermitian. However, for two colors there is an anti-unitary symmetry that can be expressed in two equivalent ways [4, 19, 51, 52],

$$[C \tau_2 K, i D(\mu)] = 0 \quad \text{with} \quad (C \tau_2 K)^2 = \mathbb{1}, \quad (2.2a)$$

$$[\gamma_5 C \tau_2 K, D(\mu)] = 0 \quad \text{with} \quad (\gamma_5 C \tau_2 K)^2 = \mathbb{1}, \quad (2.2b)$$

where  $C = i \gamma_4 \gamma_2$  is the charge conjugation matrix (see appendix A) and  $K$  is the operator of complex conjugation. Hence the Dyson index is  $\beta = 1$  in this case and we can choose

a basis in which the Dirac operator is real. This implies that  $\det D(\mu)$  is real, which also follows from  $C\tau_2\gamma_5 D(\mu)C\tau_2\gamma_5 = D(\mu)^*$  as a consequence of (2.2b).

We introduce  $N_f$  quark flavors<sup>7</sup> described by Dirac spinors  $\psi_f$  ( $f = 1, \dots, N_f$ ), which we collect in  $\psi = (\psi_1, \dots, \psi_{N_f})^T$ . Each Dirac spinor can be split into two Weyl spinors, which we collect in  $\psi_R = (\psi_{1R}, \dots, \psi_{N_f R})^T$  and  $\psi_L = (\psi_{1L}, \dots, \psi_{N_f L})^T$ . The fermionic part of the Lagrangian, including mass term and diquark sources, is given by

$$\mathcal{L}_f = \bar{\psi}[D(\mu) + MP_L + M^\dagger P_R]\psi + \frac{1}{2}\psi^T C\tau_2(J_R P_R + J_L P_L)\psi + \frac{1}{2}\psi^\dagger C\tau_2(J_R^\dagger P_R + J_L^\dagger P_L)\psi^*, \quad (2.3)$$

where  $P_{R/L} = (\mathbb{1} \pm \gamma_5)/2$  and  $M$ ,  $J_R$ , and  $J_L$  are  $N_f$ -dimensional complex matrices in flavor space. As a consequence of the Pauli principle, it suffices to take  $J_R$  and  $J_L$  to be antisymmetric since their symmetric parts drop out of the combinations in (2.3). For greater generality we have allowed for two independent matrices  $J_R$  and  $J_L$ .

We briefly summarize the symmetries of (2.3) obtained in [19, 20], assuming  $M = m\mathbb{1}$  and  $J_R = -J_L = jI$ , where  $m$  and  $j$  are real and  $I$  is defined in (A.9). For  $\mu = 0$ , the Lagrangian in the absence of mass term and diquark source is symmetric under  $SU(2N_f)$ .<sup>8</sup> The mass term breaks this symmetry to  $Sp(2N_f)$ . The diquark source transforms into the mass term under an  $SU(2N_f)$  rotation, and therefore it brings nothing new at  $\mu = 0$ . For  $\mu \neq 0$  and no sources, the chemical potential breaks the  $SU(2N_f)$  symmetry explicitly to  $SU(N_f)_R \times SU(N_f)_L \times U(1)_B$ . The mass term breaks this symmetry to  $SU(N_f)_{L+R} \times U(1)_B$ , while the diquark source breaks it to  $Sp(N_f)_R \times Sp(N_f)_L$ . In the presence of both mass term and diquark source the symmetry is broken to  $Sp(N_f)_{L+R}$ .

We now express  $\mathcal{L}_f$  in the so-called Nambu-Gor'kov formalism. Defining

$$\Psi \equiv \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}, \quad (2.4)$$

(2.3) can be rewritten as

$$\mathcal{L}_f = \frac{1}{2}\Psi^T W \Psi \quad (2.5)$$

with

$$W = \begin{pmatrix} C\tau_2(J_R P_R + J_L P_L) & -D(\mu)^T - M^T P_L - M^* P_R \\ D(\mu) + MP_L + M^\dagger P_R & -C\tau_2(J_R^\dagger P_L + J_L^\dagger P_R) \end{pmatrix}. \quad (2.6)$$

Since  $J_R$ ,  $J_L$ ,  $C$ , and  $\tau_2$  are all antisymmetric,  $W$  is antisymmetric as well, and thus

$$\int \mathcal{D}\Psi \exp\left(-\frac{1}{2} \int d^4x \Psi^T W \Psi\right) = \text{Pf}(W), \quad (2.7)$$

<sup>7</sup>We assume  $N_f < 11$  to ensure asymptotic freedom.

<sup>8</sup> $U(1)_B$  is contained in  $SU(2N_f)$ . There is an additional  $U(1)_A$  symmetry which is anomalous and therefore not considered here, but see section 5.3.



where Pf denotes the Pfaffian, which for an antisymmetric matrix of even dimension  $N$  is defined as<sup>9</sup>

$$\text{Pf}(X) \equiv \frac{1}{2^{N/2}(N/2)!} \sum_{\sigma} \text{sgn}(\sigma) X_{\sigma(1)\sigma(2)} \cdots X_{\sigma(N-1)\sigma(N)}. \quad (2.8)$$

The partition function is therefore

$$Z = \left\langle \text{Pf}(W) \right\rangle_{\text{YM}}, \quad (2.9)$$

where the subscript YM means that the average is over gauge fields weighted by the pure gauge (Yang-Mills) action. Since in this paper we will almost always work in the chiral limit we set  $M$  to zero in (2.6). As shown in appendix B.1, we then obtain from (2.9)

$$Z(J_L, J_R) = \left\langle \left[ \text{Pf}(J_R) \text{Pf}(J_L^\dagger) \right]^{n_R} \left[ \text{Pf}(J_R^\dagger) \text{Pf}(J_L) \right]^{n_L} \times \sqrt{\det'(D(\mu)^\dagger D(\mu) + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L)} \right\rangle_{\text{YM}}, \quad (2.10)$$

where  $n_R$  ( $n_L$ ) denotes the number of right- (left-) handed zero modes of  $D^\dagger D$  and the prime on the determinant means that the zero modes of  $D^\dagger D$  are omitted.<sup>10</sup> This expression is invariant under  $\text{SU}(N_f)_R \times \text{SU}(N_f)_L$ , as it should be, since  $\text{Pf}(J_R) \rightarrow \text{Pf}(U_R^T J_R U_R) = \det(U_R) \text{Pf}(J_R) = \text{Pf}(J_R)$  and likewise for  $\text{Pf}(J_L)$ .

We can also add a P- and CP-violating term  $i\theta F\tilde{F}/32\pi^2$  to the Lagrangian, where  $\tilde{F}_{\alpha\beta}^a = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta}^a$ . The topological charge  $\nu$  of a gauge field configuration is given by

$$\nu \equiv \frac{1}{32\pi^2} \int d^4x F_{\alpha\beta}^a \tilde{F}_{\alpha\beta}^a \quad (2.11)$$

and related to the number of zero modes of  $D$  by  $\nu = n_R - n_L$ .<sup>11</sup> The  $\theta$ -term corresponds, for a fixed gauge field, to a term  $i\nu\theta$  in the action. The partition function is then given by

$$Z(\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu, \quad (2.12)$$

where  $Z_\nu$  is computed by integrating only over gauge fields with fixed topology  $\nu$ .

For zero diquark sources and nonzero quark masses it is well known that an axial rotation of the quark fields changes the fermionic measure in such a way that the  $\theta$ -term can be traded for a redefinition of the quark mass matrix,  $M \rightarrow M e^{-i\theta/N_f}$  [4].<sup>12</sup> For nonzero diquark sources and in the chiral limit, it follows from (2.10) that the  $\theta$ -term can

<sup>9</sup>Later we will use the following basic properties of the Pfaffian:  $\text{Pf}(X)^2 = \det(X)$ ,  $\text{Pf}(X^T) = (-1)^{N/2} \text{Pf}(X)$ ,  $\text{Pf}(cX) = c^{N/2} \text{Pf}(X)$  for  $c \in \mathbb{C}$ , and  $\text{Pf}(UXU^T) = \det(U) \text{Pf}(X)$ .

<sup>10</sup>The numbers  $n_R$  and  $n_L$  are also equal to the number of right- and left-handed zero modes of  $D$ , see the discussion in section 3.

<sup>11</sup>Note that for  $\mu \neq 0$  the equality  $\nu = n_R - n_L$  is violated on a gauge field set of measure zero, see appendix D. Here (and also in the next two subsections) we exclude this possibility.

<sup>12</sup>In [4]  $M \rightarrow M e^{i\theta/N_f}$  is used, which is due to different conventions, see also footnotes 52 and 69.

be traded for a redefinition of the diquark sources,  $J_R \rightarrow J_R e^{i\theta_R/N_f}$  and  $J_L \rightarrow J_L e^{i\theta_L/N_f}$  with  $\theta = (\theta_R - \theta_L)/2$ .

Note that for some choices of the diquark sources and/or for a nonzero value of  $\theta$  the fermionic measure in the partition function is not positive definite, which causes some subtleties that are discussed in detail in appendix C.

## 2.2 QCD with isospin chemical potential ( $\beta = 2$ )

We now consider  $N_f = 2$  QCD at nonzero isospin chemical potential  $\mu_I = 2\mu$  for an arbitrary number of colors  $N_c \geq 3$ . The Dirac operator is given as in (2.1), except that the  $\tau_a$  are now the generators of  $SU(N_c)$  in the fundamental representation. The chemical potential  $\mu$  is assigned to the  $u$ -quark, while  $-\mu$  is assigned to the  $d$ -quark. For  $\mu \neq 0$ , the Dirac operator is no longer anti-Hermitian, but because of  $D(\mu)^\dagger = -D(-\mu)$  the fermion determinant is real and nonnegative:  $\det D(\mu)D(-\mu) = \det D(\mu)D(\mu)^\dagger \geq 0$ .<sup>13</sup> Since there is no anti-unitary symmetry we have  $\beta = 2$ .

The fermionic part of the Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \bar{u}(\gamma_\nu D_\nu + \mu\gamma_4)u + \bar{d}(\gamma_\nu D_\nu - \mu\gamma_4)d + (mu_R^\dagger u_L + md_R^\dagger d_L + \text{h.c.}) \\ &\quad + (\lambda^* u_L^\dagger d_R + \rho d_L^\dagger u_R + \text{h.c.}) \\ &= \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix} \begin{pmatrix} D(\mu) + mP_L + m^*P_R & \lambda^*P_R + \rho^*P_L \\ \rho P_R + \lambda P_L & D(-\mu) + mP_L + m^*P_R \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}, \end{aligned} \quad (2.13)$$

where  $m$  is the degenerate mass of the two quarks, and  $\rho$  and  $\lambda$  are “pionic” sources. From this we obtain (see appendix B.2)

$$Z(\rho, \lambda) = \left\langle (-\rho\lambda^*)^{n_R} (-\rho^*\lambda)^{n_L} \det'(D^\dagger D + \rho\rho^*P_R + \lambda\lambda^*P_L) \right\rangle_{\text{YM}}, \quad (2.14)$$

where the prime again indicates that the zero modes of  $D^\dagger D$  are omitted. A nonzero  $\theta$ -term can be introduced in (2.14) by redefining the pionic sources as  $\rho \rightarrow \rho e^{i\theta_R/2}$  and  $\lambda \rightarrow \lambda e^{i\theta_L/2}$ , where again  $\theta = (\theta_R - \theta_L)/2$ .

The symmetries of (2.13) with  $\rho = -\lambda \in \mathbb{R}$  are as follows. The Lagrangian at  $\mu = 0$  in the absence of sources is symmetric under  $SU(2)_L \times SU(2)_R \times U(1)_B$ , which is broken to  $SU(2) \times U(1)_B$  by the mass term or the pionic sources.<sup>14</sup> There is also a  $U(1)_A$  symmetry that is broken by the anomaly. Without sources the chemical potential breaks the  $SU(2)_L \times SU(2)_R \times U(1)_B$  symmetry to  $U(1)_L \times U(1)_R \times U(1)_B$ , where  $U(1)_{L,R}$  is generated by  $t_3 \in \mathfrak{su}(2)_{L/R}$ . The mass term breaks this symmetry to  $U(1)_{L+R} \times U(1)_B$ , while the pionic sources break it to  $U(1)_{L-R} \times U(1)_B$ . With both mass term and pionic sources the remaining symmetry is  $U(1)_B$ .

<sup>13</sup>Note that  $\det(-D^\dagger) = \det D^\dagger$  if  $D^\dagger$  has no zero modes. If it has zero modes we simply have  $0 = 0$ .

<sup>14</sup>For the mass term we end up with  $SU(2)_{L+R}$ , while for the pionic sources we end up with a different  $SU(2)$  subgroup given by the condition  $U_R^\dagger t_2 U_L = t_2$ , where  $t_2$  is the second generator of  $SU(2)$ .

### 2.3 QCD with adjoint fermions ( $\beta = 4$ )

Finally we consider QCD with fermions in the adjoint representation.<sup>15</sup> The Dirac operator in the presence of a chemical potential is given by

$$D(\mu) = \gamma_\nu D_\nu + \mu \gamma_4 \quad \text{with} \quad D_\nu = \partial_\nu \delta_{ab} + (f^c)_{ab} A_\nu^c, \quad (2.15)$$

where  $(f^c)_{ab} = f_{abc}$  denotes the generators of the adjoint representation (or structure constants). For  $\mu = 0$  the Dirac operator is anti-Hermitian, but for  $\mu \neq 0$  it again loses its Hermiticity properties. There is an anti-unitary symmetry that can be expressed in two equivalent ways [4, 19, 51, 52],

$$[CK, iD(\mu)] = 0 \quad \text{with} \quad (CK)^2 = -\mathbb{1}, \quad (2.16a)$$

$$[\gamma_5 CK, D(\mu)] = 0 \quad \text{with} \quad (\gamma_5 CK)^2 = -\mathbb{1}. \quad (2.16b)$$

Hence  $\beta = 4$  and we can choose a basis in which the Dirac operator is quaternion real. The symmetry (2.16b) implies  $C\gamma_5 D(\mu)C\gamma_5 = D(\mu)^*$ , from which it follows that  $\det D(\mu)$  is real (and actually nonnegative because all eigenvalues occur in quadruplets, see section 3.1). Because of  $(CK)^2 = -\mathbb{1}$  one can show that for  $\mu = 0$  (but not for  $\mu \neq 0$ ) the eigenvalues of the Dirac operator are twofold degenerate with linearly independent eigenstates  $\psi$  and  $C\psi^*$  (Kramers degeneracy).

Because the adjoint representation is real, it may be convenient to describe the fermions in the partition function in terms of Majorana fields. However, Majorana fermions cannot be defined in Euclidean space, and therefore we first write the Lagrangian in Minkowski space and then analytically continue to Euclidean space by a Wick rotation [4, 53]. In Minkowski space, the Lagrangian for  $N_f = 1$  Dirac fermions with diquark sources reads

$$\begin{aligned} \mathcal{L}_M^{(N_f=1)} &= \bar{\psi}(i\gamma^\nu D_\nu - m - \mu\gamma^0)\psi + \frac{1}{2}[\psi^T C(j_R P_R + j_L P_L)\psi + \text{h.c.}] \\ &= \frac{1}{2} \begin{pmatrix} \bar{\psi}^T \\ \psi \end{pmatrix}^T \begin{pmatrix} -C(j_R^* P_L + j_L^* P_R) & i\gamma^\nu D_\nu - m - \mu\gamma^0 \\ C(-i\gamma^\nu D_\nu + m - \mu\gamma^0) & C(j_R P_R + j_L P_L) \end{pmatrix} \begin{pmatrix} \bar{\psi}^T \\ \psi \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \psi_c \\ \psi \end{pmatrix}^T \begin{pmatrix} C(j_R^* P_L + j_L^* P_R) & -C(i\gamma^\nu D_\nu - m - \mu\gamma^0) \\ C(-i\gamma^\nu D_\nu + m - \mu\gamma^0) & C(j_R P_R + j_L P_L) \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi \end{pmatrix}, \end{aligned} \quad (2.17)$$

where in the last line we have defined  $\psi_c = C\bar{\psi}^T$ . The partition function is thus given by

$$Z_M^{(N_f=1)} = \left\langle \text{Pf} \begin{pmatrix} C(j_R^* P_L + j_L^* P_R) & -C(i\gamma^\nu D_\nu - m - \mu\gamma^0) \\ C(-i\gamma^\nu D_\nu + m - \mu\gamma^0) & C(j_R P_R + j_L P_L) \end{pmatrix} \right\rangle_{\text{YM}}. \quad (2.18)$$

This result generalizes trivially to general  $N_f$ . The partition function is then simply the expectation value of a product of  $N_f$  Pfaffians, possibly with different masses.

For simplicity we now take the chiral limit. After Wick rotation, the partition function for  $N_f$  flavors in Euclidean space becomes

$$Z_E = \left\langle \text{Pf} \begin{pmatrix} C(J_R^* P_L + J_L^* P_R) & CD(\mu) \\ -CD(\mu)^\dagger & C(J_R P_R + J_L P_L) \end{pmatrix} \right\rangle_{\text{YM}}, \quad (2.19)$$

<sup>15</sup>The results obtained for the adjoint representation easily generalize to other real representations.

where  $J_L$  and  $J_R$  are now symmetric matrices of dimension  $N_f$ . For  $\beta = 4$ ,  $N_f$  does not have to be even. In appendix B.3 we show that this leads to

$$Z(J_L, J_R) = \left\langle \det(-J_R J_L^\dagger)^{n_R/2} \det(-J_R^\dagger J_L)^{n_L/2} \det''(D^\dagger D + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L) \right\rangle_{\text{YM}}, \quad (2.20)$$

where  $\det''$  indicates that the zero modes of  $D^\dagger D$  are omitted and that each degenerate eigenvalue of  $D^\dagger D$  is counted only once. It follows from the derivation in appendix B.3 that a nonzero  $\theta$ -term can be introduced in (2.20) by redefining  $J_R \rightarrow J_R e^{i\theta_R/2N_f N_c}$  and  $J_L \rightarrow J_L e^{i\theta_L/2N_f N_c}$ , where again  $\theta = (\theta_R - \theta_L)/2$ .

The symmetries of (2.17) extended to general  $N_f$  with degenerate masses and  $J_R = -J_L = j\mathbb{1}$  with real  $j$  are as follows [20]. The Lagrangian at  $\mu = 0$  in the absence of sources is symmetric under  $\text{SU}(2N_f)$ , which is broken to  $\text{SO}(2N_f)$  by the quark mass or the diquark source.<sup>16</sup> There is also the usual anomalous  $\text{U}(1)_A$  symmetry. Without sources the chemical potential breaks the  $\text{SU}(2N_f)$  symmetry to  $\text{SU}(N_f)_L \times \text{SU}(N_f)_R \times \text{U}(1)_B$ . The mass term breaks this symmetry to  $\text{SU}(N_f)_{L+R} \times \text{U}(1)_B$ , while the diquark source breaks it to  $\text{SO}(N_f)_L \times \text{SO}(N_f)_R$ . With both mass term and diquark source the remaining symmetry is  $\text{SO}(N_f)_{L+R}$ .

### 3 Eigenvalues and singular values of the Dirac operator

In this section we discuss the eigenvalues and singular values of the Dirac operator and related quantities. Two preliminary remarks are in order.

First, some parts of the discussion rely on the index theorem. In appendix D we show that for a non-Hermitian Dirac operator such as  $D(\mu)$  the index theorem takes the form

$$\frac{1}{32\pi^2} \int d^4x F \tilde{F} = \frac{1}{2} [\text{ind } D(\mu) + \text{ind } D(\mu)^\dagger] \quad (3.1)$$

and that

$$\text{ind } D(\mu) = \text{ind } D(\mu)^\dagger \quad \text{almost surely}, \quad (3.2)$$

where  $\text{ind } D(\mu) = \dim \ker D_R - \dim \ker D_L$ , see (D.1) for the notation. The meaning of “almost surely” is that to have  $\text{ind } D(\mu) \neq \text{ind } D(\mu)^\dagger$  the gauge field needs to be fine-tuned, which corresponds to a gauge field set of measure zero. In this section we ignore this set of measure zero and use the index theorem in its usual form.

Second, we implicitly assume a regularization (such as lattice QCD) that allows us to count the number of eigenstates. Some of our arguments rely on the index theorem, which can be (and usually is) violated by the regulator. Therefore our results apply only after the regulator has been removed. We assume that the procedure of removing the regulator does not invalidate the results that rely on the index theorem.

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<sup>16</sup>As in the case of  $\beta = 1$ , the diquark source transforms into the mass term under an  $\text{SU}(2N_f)$  rotation.

### 3.1 Eigenvalues

The discussion of this section extends that of [52]. The eigenvalue equation for the Dirac operator is

$$D(\mu)\psi_n = \lambda_n\psi_n. \quad (3.3)$$

For simplicity we will omit the argument  $\mu$  if no confusion is likely to arise.

Let us first consider  $\mu = 0$ .<sup>17</sup> In that case the eigenvalues  $\lambda_n$  are purely imaginary, and because of  $\{D, \gamma_5\} = 0$  the nonzero eigenvalues come in pairs  $\pm\lambda_n$ . If the eigenstate corresponding to  $\lambda_n$  is  $\psi_n$ , then the eigenstate corresponding to  $-\lambda_n$  is  $\gamma_5\psi_n$ . There can also be eigenvalues equal to zero, and the corresponding eigenstates can be arranged to be eigenstates of  $\gamma_5$ . In general there are  $n_R$  ( $n_L$ ) right-handed (left-handed) zero modes with eigenvalue  $+1$  ( $-1$ ) of  $\gamma_5$ . The difference  $n_R - n_L$  is equal to the topological charge  $\nu$  of the underlying gauge field configuration via the index theorem and stable under perturbations of the gauge field. All other possible zero modes are accidental in the sense that they require a fine-tuning of the gauge field. This implies that generically there are only zero modes of one chirality. In the remainder of this section we assume that there are no accidental zero modes.

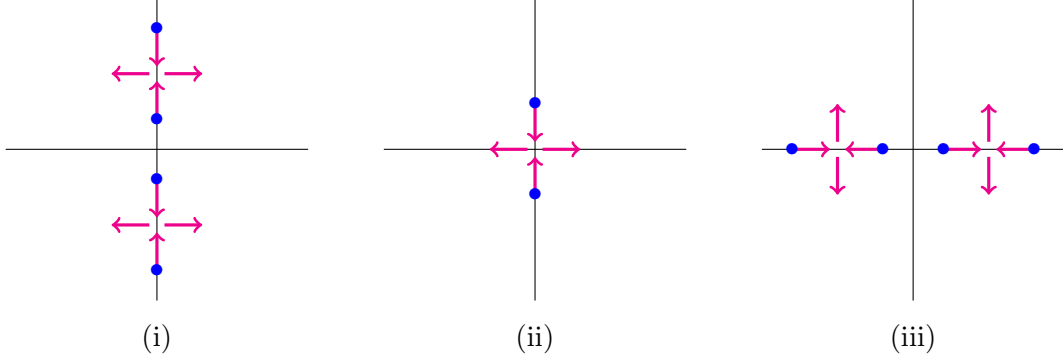
We now discuss what happens to the eigenvalues as  $\mu$  is turned on (for a fixed gauge field and in a finite volume) and first consider  $\beta = 1$  (e.g., two-color QCD). Since  $D(\mu \neq 0)$  is no longer anti-Hermitian one would expect the eigenvalues to move into the complex plane for any  $\mu$ . However, using the symmetry (2.2b) one can show that the nonzero eigenvalues come either in quadruplets  $\lambda, -\lambda, \lambda^*, -\lambda^*$  (with eigenstates  $\psi, \gamma_5\psi, C\tau_2\gamma_5\psi^*, C\tau_2\psi^*$ ) if  $\lambda$  is complex or in pairs  $\pm\lambda$  (with eigenstates  $\psi, \gamma_5\psi$ ) if  $\lambda$  is purely real or purely imaginary. Since at  $\mu = 0$  the nonzero eigenvalues are generically nondegenerate (because of level repulsion due to interactions), a quadruplet cannot be formed for infinitesimally small  $\mu$ .<sup>18</sup> What happens (see figure 1) is that as a function of  $\mu$ , the eigenvalues move along the imaginary axis until two eigenvalues (and their partners) become degenerate, i.e., the effect of  $\mu$  overcomes the level repulsion. As  $\mu$  is increased further, these four eigenvalues move into the complex plane and form a quadruplet. Another possibility is that a pair of originally imaginary eigenvalues hits zero and then becomes a pair of real eigenvalues. Finally, two real eigenvalues (and their partners) can also merge and then become a quadruplet.<sup>19</sup> The original topological eigenvalues equal to zero (but not the accidental ones) are stable under the perturbation by  $\mu$ . The corresponding zero modes change smoothly with  $\mu$  and remain eigenstates of  $\gamma_5$  [54–57].

For  $\beta = 2$  there is no anti-unitary symmetry, and the eigenvalues (which still come in pairs  $\pm\lambda$  with eigenstates  $\psi, \gamma_5\psi$ ) move into the complex plane immediately as  $\mu$  is switched on. Exactly real or imaginary eigenvalues only occur accidentally, i.e., if the gauge field is fine-tuned. The topological zero modes behave as in the case of  $\beta = 1$ .

<sup>17</sup>The arguments for  $\mu = 0$  apply to all values of the Dyson index  $\beta$ , except that for  $\beta = 4$  we have the additional Kramers degeneracy already mentioned in section 2.3.

<sup>18</sup>The assumption of a finite volume is essential here.

<sup>19</sup>All of these three possibilities have been verified in lattice simulations and random matrix studies. We thank Jacques Bloch for performing the lattice simulations.



**Figure 1.** Flow of the Dirac eigenvalues for a fixed gauge field as a function of  $\mu$  for  $\beta = 1$ . (i) Two imaginary eigenvalues (and their partners) merge and move into the complex plane to form a quadruplet. (ii) A pair of imaginary eigenvalues merges at zero and becomes a pair of real eigenvalues. (iii) Two real eigenvalues (and their partners) merge and move into the complex plane to form a quadruplet.

For  $\beta = 4$  it follows from the symmetry (2.16b) that the nonzero eigenvalues come in quadruplets  $\lambda, -\lambda, \lambda^*, -\lambda^*$  (with eigenstates  $\psi, \gamma_5\psi, C\gamma_5\psi^*, C\psi^*$ ) if  $\lambda$  is complex and that the Kramers degeneracy is removed once  $\mu$  is switched on. Hence we do not need the mechanism of figure 1(i) and the eigenvalues move into the complex plane immediately. As in the case of  $\beta = 2$ , exactly real or imaginary eigenvalues only occur accidentally. The topological zero modes are twofold degenerate for both  $\mu = 0$  and  $\mu \neq 0$ . They change smoothly with  $\mu$  and are eigenstates of  $\gamma_5$  with the same chirality.

### 3.2 Singular values

Let us now consider the operator  $D^\dagger D$ . Since this operator is Hermitian with nonnegative eigenvalues, we write its eigenvalue equation in the form

$$D^\dagger D \varphi_n = \xi_n^2 \varphi_n. \quad (3.4)$$

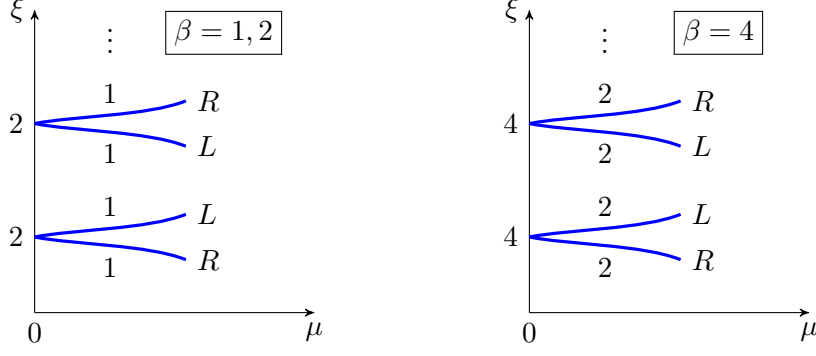
The  $\xi_n$  are real and nonnegative. They are called the singular values of  $D$ , the name coming from the singular value decomposition of a non-Hermitian matrix.<sup>20</sup> The operators  $D^\dagger D$  and  $DD^\dagger$  share all nonzero eigenvalues, since (3.4) implies

$$DD^\dagger (D\varphi_n) = \xi_n^2 (D\varphi_n), \quad (3.5)$$

and similarly the other way around.

At  $\mu = 0$  the  $\lambda_n$  and  $\xi_n$  are trivially related by  $\xi_n = |\lambda_n|$ , and therefore the nonzero singular values are twofold degenerate (for  $\beta = 1$  and 2) or fourfold degenerate (for  $\beta = 4$ ). At  $\mu \neq 0$  there is no simple relation between the  $\lambda_n$  and  $\xi_n$ , i.e., knowing only the  $\lambda_n$  we cannot compute the  $\xi_n$ , and vice versa. As soon as  $\mu$  is turned on the twofold degeneracy (for  $\beta = 1$  and 2) is removed, and the fourfold degeneracy (for  $\beta = 4$ ) is reduced to a twofold degeneracy, see the argument after (B.33).

<sup>20</sup>In this paper we assume that the extension of the singular value decomposition to non-Hermitian operators is straightforward and skip the mathematical foundations.



**Figure 2.** Schematic flow of the singular values of  $D$  for a fixed gauge field as a function of  $\mu$  for the three symmetry classes. The numbers stand for the degeneracy of each singular value.

The operator  $D^\dagger D$  commutes with  $\gamma_5$ , and therefore the states  $\varphi_n$  have definite chirality,  $\gamma_5 \varphi_n = \pm \varphi_n$  (or can be so arranged if the singular values are degenerate). Now assume  $\xi_n > 0$  and define  $\tilde{\varphi}_n = \xi_n^{-1} D \varphi_n$ , for which

$$DD^\dagger \tilde{\varphi}_n = \xi_n^{-1} DD^\dagger D \varphi_n = \xi_n D \varphi_n = \xi_n^2 \tilde{\varphi}_n, \quad (3.6)$$

$$\gamma_5 \tilde{\varphi}_n = \xi_n^{-1} \gamma_5 D \varphi_n = -\xi_n^{-1} D \gamma_5 \varphi_n = \mp \tilde{\varphi}_n, \quad (3.7)$$

i.e.,  $\tilde{\varphi}_n$  is an eigenstate of  $DD^\dagger$  with the same eigenvalue but chirality opposite to that of  $\varphi_n$ . Therefore the number of right-handed (left-handed) nonzero modes of  $D^\dagger D$  equals the number of left-handed (right-handed) nonzero modes of  $DD^\dagger$ . Since these operators coincide at  $\mu = 0$ , the number of right-handed and left-handed nonzero modes of  $D^\dagger D$  is equal at  $\mu = 0$ . As  $\mu$  is turned on from zero, a right-handed (or left-handed) nonzero mode of  $D^\dagger D$  changes its form smoothly but stays right-handed (or left-handed) since the eigenvalue of  $\gamma_5$  is discrete. Therefore the numbers of right-handed and left-handed nonzero modes of  $D^\dagger D$  and  $DD^\dagger$  are all equal (assuming that there are no accidental zero modes).

If  $D$  has an eigenvalue equal to zero,  $D\psi = 0$  trivially implies  $D^\dagger D\psi = 0$ , i.e., there is a corresponding singular value of  $D$  equal to zero, and the zero mode of  $D^\dagger D$  is the same as that of  $D$ . If  $\psi$  is not a zero mode of  $D$ , it cannot be a zero mode of  $D^\dagger D$  either since  $\langle \psi | D^\dagger D | \psi \rangle \neq 0$ . As discussed in section 3.1, all zero modes of  $D$  and therefore of  $D^\dagger D$  generically have the same chirality. Using  $D(\mu)^\dagger = -D(-\mu)$  and our earlier observation that the number of zero modes is stable as a function of  $\mu$ , we conclude that (i) the operator  $DD^\dagger$  has the same number of zero modes as  $D^\dagger D$ , (ii) the zero modes of  $DD^\dagger$  are equal to those of  $D(-\mu)$ , and (iii) the chirality of the zero modes of  $D^\dagger D$  and  $DD^\dagger$  is the same.<sup>21</sup>

In figure 2 we schematically illustrate the singular value flow in QCD-like theories with  $\beta = 1, 2$ , and 4. All eigenstates of  $D^\dagger D$  have definite chirality, as indicated by  $R$  and  $L$  in the figure. For  $\beta = 4$  the zero modes of  $D$ , and therefore of  $D^\dagger D$ , are twofold degenerate for both  $\mu = 0$  and  $\mu \neq 0$ .

<sup>21</sup>Conclusion (iii) is not necessarily valid if there are accidental zero modes, while (i) and (ii) are always valid, see appendix D.

### 3.3 Dirac operator with chiral chemical potential

In this subsection we derive a relation between the singular values and the eigenvalues of the Dirac operator with chiral (or axial) chemical potential defined by

$$D_5(\mu) = \gamma_\nu D_\nu + \mu \gamma_4 \gamma_5. \quad (3.8)$$

This operator was introduced in the context of the chiral magnetic effect [58].

Since  $D_5(\mu)$  is anti-Hermitian its eigenvalues are purely imaginary, and since it anti-commutes with  $\gamma_5$  its nonzero eigenvalues come in pairs with opposite sign. Now, for any right-handed (left-handed) spinor  $\varphi_R$  ( $\varphi_L$ ) we have

$$\begin{aligned} D_5(\mu)^2 \varphi_R &= (\gamma_\nu D_\nu + \mu \gamma_4 \gamma_5)(\gamma_\nu D_\nu + \mu \gamma_4 \gamma_5) \varphi_R \\ &= (\gamma_\nu D_\nu + \mu \gamma_4 \gamma_5)(\gamma_\nu D_\nu + \mu \gamma_4) \varphi_R \\ &= (\gamma_\nu D_\nu - \mu \gamma_4)(\gamma_\nu D_\nu + \mu \gamma_4) \varphi_R \\ &= -D(\mu)^\dagger D(\mu) \varphi_R \end{aligned} \quad (3.9)$$

and similarly

$$D_5(\mu)^2 \varphi_L = -D(\mu) D(\mu)^\dagger \varphi_L, \quad (3.10)$$

$$D_5(-\mu)^2 \varphi_R = -D(\mu) D(\mu)^\dagger \varphi_R, \quad (3.11)$$

$$D_5(-\mu)^2 \varphi_L = -D(\mu)^\dagger D(\mu) \varphi_L, \quad (3.12)$$

which can also be expressed in terms of the decompositions

$$D_5(\mu)^2 = -D^\dagger D P_R - D D^\dagger P_L, \quad (3.13)$$

$$D_5(-\mu)^2 = -D^\dagger D P_L - D D^\dagger P_R, \quad (3.14)$$

or equivalently

$$D^\dagger D = -D_5(\mu)^2 P_R - D_5(-\mu)^2 P_L, \quad (3.15)$$

$$D D^\dagger = -D_5(\mu)^2 P_L - D_5(-\mu)^2 P_R. \quad (3.16)$$

Let us denote the singular values corresponding to the right-handed (left-handed) nonzero modes of  $D^\dagger D$  by  $\xi_{Rn}$  ( $\xi_{Ln}$ ). From the above arguments we conclude that the nonzero eigenvalues of  $D_5(\mu)$  and  $D_5(-\mu)$  are given by the sets  $\{\pm i \xi_{Rn}\}$  and  $\{\pm i \xi_{Ln}\}$ , respectively.

Disregarding accidental zero modes, the zero modes of  $D^\dagger D$  and  $D D^\dagger$  have only one chirality. If they are all right-handed, the zero modes of  $D_5(\mu)$  ( $D_5(-\mu)$ ) are right-handed and given by those of  $D^\dagger D$  ( $D D^\dagger$ ). If they are all left-handed, the zero modes of  $D_5(\mu)$  ( $D_5(-\mu)$ ) are left-handed and given by those of  $D D^\dagger$  ( $D^\dagger D$ ).

Since  $D_5(\mu)^\dagger = -D_5(\mu)$  and  $\{D_5(\mu), \gamma_5\} = 0$ , the Banks-Casher relation for the chiral condensate can be extended to nonzero chiral chemical potential without difficulty,

$$\langle \bar{\psi} \psi \rangle = \pi \rho_5(0), \quad (3.17)$$

where  $\rho_5(\lambda)$  is the spectral density of  $D_5(\mu)$  with fermionic weight  $\det^{N_f} D_5(\mu)$ .



## 4 Banks-Casher-type relations

In this section we derive Banks-Casher-type relations for the three different theories with  $\beta = 1, 2$ , and  $4$ . In each case, a particular condensate is related to the density of the singular values at the origin. For our derivations to be correct it is important that in the computation of the partition function (including source term for the desired condensate) the fermionic measure is positive definite. If this requirement is not met a probabilistic interpretation is not possible and a number of subtleties arise that are discussed in some detail in appendix C. In the present section we restrict ourselves to cases in which the measure is positive definite.

### 4.1 Two-color QCD ( $\beta = 1$ )

For simplicity we take  $J_R = j_R I$  and  $J_L = j_L I$  with  $I$  given in (A.9) and numbers  $j_R, j_L$  that for the time being can be complex. It follows from (2.10) and the discussion after (2.12) that the measure is positive definite only if the combination  $e^{i\theta}(-j_R j_L^*)^{N_f/2}$  is real and positive. If we assume real sources and  $\theta = 0$ , this condition is always satisfied for  $j_R = -j_L$  (corresponding to the scalar diquark condensate). If  $N_f/2$  is odd, it is violated for  $j_R = j_L$  (corresponding to the pseudoscalar diquark condensate). In the following we therefore set  $j_R = -j_L = j$  with  $j$  real,<sup>22</sup> which implies  $J_L^\dagger J_L P_L + J_R^\dagger J_R P_R = j^2 \mathbb{1}_{N_f}$ . From (2.10) we then obtain

$$Z(j) = \left\langle \det^{N_f/2}(D^\dagger D + j^2) \right\rangle_{\text{YM}} = \left\langle \prod_n (\xi_n^2 + j^2)^{N_f/2} \right\rangle_{\text{YM}}. \quad (4.1)$$

Introducing the notation

$$\langle O \rangle_j = \frac{1}{Z(j)} \left\langle O \det^{N_f/2}(D^\dagger D + j^2) \right\rangle_{\text{YM}} \quad (4.2)$$

for expectation values in the presence of a diquark source, we define the density of the nonzero<sup>23</sup> singular values of the Dirac operator by

$$\rho_{\text{sv}}(\xi) = \lim_{V_4 \rightarrow \infty} \frac{1}{V_4} \left\langle \sum_n \delta(\xi - \xi_n) \right\rangle_{j=0} \quad \text{for } \xi > 0, \quad (4.3)$$

where  $V_4$  is the space-time volume. The scalar diquark condensate can then be expressed in terms of this density at the origin,

$$\begin{aligned} \langle \psi^T C \gamma_5 \tau_2 I \psi \rangle &= \lim_{j \rightarrow 0^+} \lim_{V_4 \rightarrow \infty} \frac{1}{V_4} \frac{\partial}{\partial j} \ln Z(j) \\ &= \lim_{j \rightarrow 0^+} \lim_{V_4 \rightarrow \infty} \frac{1}{V_4} \frac{N_f}{2} \left\langle \sum_n \frac{2j}{\xi_n^2 + j^2} \right\rangle_j \\ &= \frac{N_f}{2} \int_0^\infty d\xi \rho_{\text{sv}}(\xi) \lim_{j \rightarrow 0^+} \frac{2j}{\xi^2 + j^2} \\ &= \frac{N_f}{2} \pi \rho_{\text{sv}}(0), \end{aligned} \quad (4.4)$$

<sup>22</sup>We will comment on the case  $j_R = j_L$  below.

<sup>23</sup>See appendix C.2 for a discussion of the zero-mode contribution.

which is similar to the Banks-Casher relation for the chiral condensate in terms of the Dirac eigenvalue density at zero. Our analysis can be extended to nonzero quark masses. Assuming for simplicity  $M = m\mathbb{1}$  with real  $m$ , we again obtain (4.1), but with  $D$  and  $D^\dagger$  replaced by  $D + m$  and  $D^\dagger + m$ . This means that (4.4) continues to hold, except that the singular values are now those of  $D + m$ .

Some comments are in order. First, (4.4) holds at  $\mu = 0$  and  $\mu \neq 0$ , whereas the original Banks-Casher relation only holds at  $\mu = 0$ . Second, in the derivation of (4.4) we have tacitly dropped the contribution of the singular values equal to zero. As discussed in appendix C, this is only justified if the measure is positive definite. Third, the integral in the third line of (4.4) needs a proper UV regularization. This was discussed carefully for the original Banks-Casher relation in [4] and works in exactly the same way here. The point is that the UV-divergent part disappears in the limit  $j \rightarrow 0^+$ . Fourth, we observe that setting  $j_R = j_L = j$  we would have obtained  $\det W = \det(D^\dagger D + j^2)$  just as in (4.1), which formally would have led to (4.4) but with the pseudoscalar diquark condensate  $\langle \psi^T C \tau_2 I \psi \rangle$  on the l.h.s. instead. However, for odd  $N_f/2$  the measure is not positive definite for  $j_R = j_L$ , which invalidates the Banks-Casher relation for the pseudoscalar condensate, see appendix C.2. For even  $N_f/2$  the measure is positive for both types of sources, and what condensate is given by  $\rho_{\text{sv}}(0)$  depends on the choice of sources we add.<sup>24</sup> In appendix C.3 we show that this is not in contradiction with QCD inequalities. Finally, we note that for two flavors and in the chiral limit, the relation (4.4) was obtained earlier by Fukushima [49], see also [59, 60]. Our result differs from that of [49] by a factor of 1/2. The contribution of the zero modes and the positive definiteness of the measure were not addressed in [49].

At  $\mu = 0$ , the spectra of the eigenvalues and singular values are identical, and so are the spectral densities at the origin. This implies that the chiral condensate (for  $m \rightarrow 0$ ) and the diquark condensate (for  $j \rightarrow 0$ ) are of the same magnitude, which is consistent with the fact that under a global  $\text{SU}(2N_f)$  rotation these condensates can be rotated into each other. As is well known,  $\mu \neq 0$  breaks this degeneracy. As  $\mu$  increases (for  $m \neq 0$ ), the diquark condensate remains exactly zero until a critical value  $\mu_c = m_\pi/2$  is reached, and then starts growing for  $\mu > \mu_c$  [20]. This behavior can be naturally understood by our Banks-Casher-type relation. For  $m \neq 0$  the relation reads  $\langle \psi \psi \rangle \propto \rho_{\text{sv}}(0; m)$ , where  $\rho_{\text{sv}}(\lambda; m)$  stands for the singular value density of  $D(\mu) + m$ , as remarked below (4.4). For sufficiently small  $\mu$ , all eigenvalues of  $D(\mu)$  are still localized near the imaginary axis, and the density of the near-zero modes of  $D(\mu) + m$  is zero. As  $\mu$  increases, the eigenvalues spread out more, and for  $\mu > \mu_c$  there is a nonzero density of eigenvalues at  $\pm m$ . This signals a nonzero  $\rho_{\text{sv}}(0; m)$ , i.e., the onset of diquark condensation.<sup>25</sup>

It is also possible to express the partition function in terms of the Dirac operator with chiral chemical potential  $D_5(\mu)$  defined in (3.8). This is most easily shown working

<sup>24</sup>More precisely, the magnitude of the condensate is given by  $\rho_{\text{sv}}(0)$  and its orientation by the external sources. A simple analog is the Ising model, where the direction of the spontaneous magnetization at zero temperature depends on the direction of the (infinitesimal) external magnetic field. See also appendix C.

<sup>25</sup>A similar discussion starting from a different method can be found in [61].

backwards starting from (4.1),

$$\begin{aligned} Z(j) &= \left\langle \prod_n (\xi_n^2 + j^2)^{N_f/2} \right\rangle_{\text{YM}} = \left\langle j^{N_f|\nu|} \left[ \prod'_n (\xi_{Rn}^2 + j^2) \prod'_n (\xi_{Ln}^2 + j^2) \right]^{N_f/2} \right\rangle_{\text{YM}} \\ &= \left\langle \det^{N_f/2}(D_5(\mu) + j) \det^{N_f/2}(D_5(-\mu) + j) \right\rangle_{\text{YM}}, \end{aligned} \quad (4.5)$$

where  $|\nu|$  is the number of topological zero modes of  $D^\dagger D$ , the primed products are only over nonzero singular values, and in the last line we have used the relationships between the eigenvalues of  $D_5$  and the singular values derived in section 3.3.

#### 4.2 QCD with isospin chemical potential ( $\beta = 2$ )

It follows from (2.14) that the measure is positive definite only if the combination  $e^{i\theta}(-\rho\lambda^*)$  is real and positive. We therefore choose  $\theta = 0$  and  $\rho = -\lambda = j$  with real  $j$  (another choice for which the measure is not positive definite is considered in appendix C.1). The partition function is then

$$Z(j) = \left\langle \det(D^\dagger D + j^2) \right\rangle_{\text{YM}}, \quad (4.6)$$

and by a calculation analogous to section 4.1 we obtain the pion condensate

$$\langle \bar{u}\gamma_5 d - \bar{d}\gamma_5 u \rangle = \pi \rho_{\text{sv}}(0). \quad (4.7)$$

Similar comments as in section 4.1 apply. In particular, we have dropped the contributions of the zero modes (which is justified because the measure is positive definite), and a proper UV regularization is understood. As in section 4.1 we can also express the partition function in terms of  $D_5(\mu)$  and obtain the same result as in (4.5).

#### 4.3 QCD with adjoint fermions ( $\beta = 4$ )

We now take  $J_R = j_R \mathbb{1}$  and  $J_L = j_L \mathbb{1}$ . The measure in (2.20) is then positive definite only if the combination  $e^{i\theta/N_c}(-j_R j_L^*)^{N_f}$  is real and positive. As in section 4.1 we therefore assume  $\theta = 0$  and set  $j_R = -j_L = j$  with real  $j$  (see appendix C.1 for another choice). The partition function is then

$$Z(j) = \left\langle \det^{N_f/2}(D^\dagger D + j^2) \right\rangle_{\text{YM}}, \quad (4.8)$$

and a calculation analogous to section 4.1 leads to

$$\langle \psi^T C \gamma_5 \psi \rangle = \frac{N_f}{2} \pi \rho_{\text{sv}}(0). \quad (4.9)$$

Again, similar comments as in section 4.1 apply, and we could also have expressed the partition function as in (4.5).

### 5 Chiral Lagrangians with diquark source ( $\beta = 1$ )

From now on we concentrate on two-color QCD ( $\beta = 1$ ) for simplicity and brevity. Results similar to those obtained in sections 5–7 could also be derived for the theories with  $\beta = 2$  and  $\beta = 4$  using the methods employed here, but we will not pursue this here.

### 5.1 Three different regimes: low, intermediate, and high density

We now construct the low-energy effective theory for two-color QCD at nonzero density in the presence of a diquark source and in the chiral limit. There are actually three different effective theories, applicable at low, intermediate, and high density. In the following we will refer to them as **L**, **I**, and **H**, respectively. The effective theory **L** is constructed under the assumption of maximal chiral symmetry breaking at low density [19, 20], while the effective theory **I** is constructed assuming the conjectured BEC-BCS crossover discussed in the introduction.<sup>26</sup> On the other hand, the effective theory **H** is constructed based on a rigorous weak-coupling analysis at high density [15].

In this subsection we discuss how the three density regimes differ in their patterns of spontaneous symmetry breaking and the number of Nambu-Goldstone (NG) modes and comment on the connection between the three regimes. The effective theories themselves and mass formulas for the NG modes will then be derived in the next three subsections, and their domains of validity will be discussed in section 5.5.

At very low density one can start from the pattern of chiral symmetry breaking at *zero* density,

$$\mathrm{SU}(2N_f) \rightarrow \mathrm{Sp}(2N_f), \quad (5.1)$$

and treat the chemical potential and the diquark source as a small perturbation. This is the approach taken in [19, 20]. The NG modes of the theory are the same as those of the zero-density theory, i.e., they are collected in a field  $\Sigma$  that parametrizes the coset space  $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$ .  $\mathrm{SU}(N)$  and  $\mathrm{Sp}(N)$  have  $N^2 - 1$  and  $N(N + 1)/2$  generators, respectively, hence the number of NG modes in this regime is  $N_f(2N_f - 1) - 1$ . One should keep in mind, however, that some of these modes acquire a mass as  $\mu$  is increased.

At intermediate density, when  $\mu$  can no longer be treated as a small perturbation, we first recall that  $\mu$  breaks the original  $\mathrm{SU}(2N_f)$  symmetry to  $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B$ . A diquark condensate then breaks this symmetry to  $\mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$  so that the symmetry-breaking pattern is now

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \rightarrow \mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R. \quad (5.2)$$

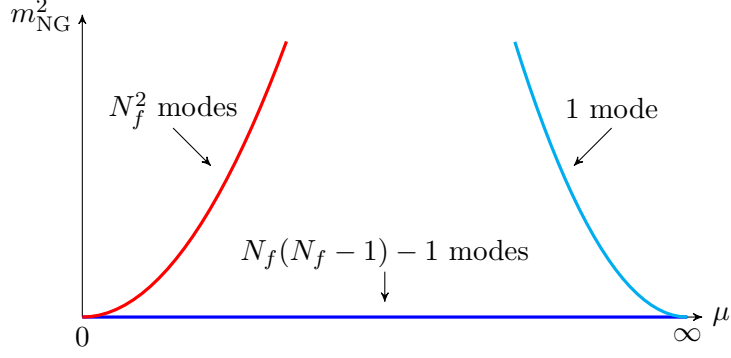
The corresponding NG modes are  $\Sigma_L, \Sigma_R \in \mathrm{SU}(N_f)/\mathrm{Sp}(N_f)$  and  $V \in \mathrm{U}(1)$ , and the total number of NG modes in this regime is  $N_f(N_f - 1) - 1$ .

At very high density the  $\mathrm{U}(1)_A$  anomaly is suppressed due to the screening of instantons [66–68]. We therefore need to take the original  $\mathrm{U}(1)_A$  symmetry of the action into account. It is no longer broken explicitly by the anomaly but spontaneously by the diquark condensate so that the symmetry-breaking pattern is

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A \rightarrow \mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R. \quad (5.3)$$

---

<sup>26</sup>From this point of view, our results from the effective theory **I** below should be viewed as predictions to be verified in future lattice simulations, which would then (dis)confirm the conjectured BEC-BCS crossover. Other possibilities for effective theories in the intermediate-density regime have been considered earlier, see, e.g., [62, 63]. In the model of [62] certain vector mesons could become massless at nonzero density, while in [63] this does not happen. Lattice studies on this issue are inconclusive [64, 65]. Here, we assume that all vector mesons remain massive at all densities.



**Figure 3.** Schematic  $\mu$ -dependence of the masses of the NG modes. See text for details.

The NG modes are the same as at intermediate density, except that there is an additional NG mode corresponding to  $U(1)_A$  which we call  $\eta'$ . In other words, the  $\eta'$  mass has become negligible. Hence the total number of NG modes in this regime is  $N_f(N_f - 1)$ .

Let us make some qualitative comments on the connection between the three regimes, starting at zero density. Assuming the diquark sources to be infinitesimally small, there are  $N_f(2N_f - 1) - 1$  massless NG modes at  $\mu = 0$ . As  $\mu$  is increased,  $N_f(N_f - 1) - 1$  of them remain massless while  $N_f^2$  of them acquire a  $\mu$ -dependent mass [20]. Starting from the effective theory at zero density, these  $N_f^2$  modes can be integrated out. This yields the effective theory at intermediate density, which does not depend explicitly on  $\mu$  but contains parameters (i.e., low-energy constants) that have acquired a  $\mu$ -dependence through the integrating-out of the massive modes. A similar argument applies starting at high density. As  $\mu$  is lowered, the  $U(1)_A$  anomaly reappears so that the  $\eta'$  becomes massive and can be integrated out. These comments are illustrated in figure 3 and will be made more quantitative in the next subsections.

Before proceeding, let us discuss a somewhat subtle issue regarding the parity of the diquark condensate. In general, when constructing an effective theory, one starts from an *assumption* of how the symmetries of the theory are broken, i.e., this assumption is an input to the effective theory. In the construction of the effective theories below, one assumption we have to put in is whether the scalar or the pseudoscalar diquark condensate minimizes the ground-state energy. For nonzero mass and zero diquark source it was shown by QCD inequalities that if a diquark condensate forms, it does so in the scalar channel [19]. However, for zero mass and nonzero diquark source QCD inequalities do not provide any information on this question, see appendix C.3. The assumption we will make is that the diquark condensate again forms in the scalar channel. This assumption is based on instanton dynamics at high density, see, e.g., the review [69]. The instanton vertex is  $c \det(\psi_L^\dagger \psi_R) + \text{h.c.}$  with  $c > 0$ , where we have  $N_f$  legs each for  $\psi_R$  and  $\psi_L$  [70]. Taking the expectation value with respect to the diquark-condensed ground state, the contribution of the instanton vertex to the energy is  $c' \langle \psi_L^\dagger \psi_L \rangle^{N_f/2} \langle \psi_R \psi_R \rangle^{N_f/2}$  with  $c' > 0$ . For  $N_f = 4n + 2$  with  $n \in \mathbb{N}$  we obtain  $c' (\langle \psi_L^\dagger \psi_L \rangle \langle \psi_R \psi_R \rangle)^{2n} \times \langle \psi_L^\dagger \psi_L \rangle \langle \psi_R \psi_R \rangle$ , which is negative for  $\langle \psi_L \psi_L \rangle = -\langle \psi_R \psi_R \rangle$  but positive for  $\langle \psi_L \psi_L \rangle = \langle \psi_R \psi_R \rangle$ . Therefore, the positive-parity state is favored by instantons. We assume that this argument carries

over to low density based on the conjectured BEC-BCS crossover. For  $N_f = 4n$  the argument does not apply since the contribution to the energy is  $c'(\langle\psi_L^\dagger\psi_L^\dagger\rangle\langle\psi_R\psi_R\rangle)^{2n}$ , which is not affected by a relative sign between  $\langle\psi_L\psi_L\rangle$  and  $\langle\psi_R\psi_R\rangle$ . Although in the construction of the effective theories below we do not distinguish between  $N_f = 4n$  and  $N_f = 4n + 2$ , we will see that for  $N_f = 4n$  the positive- and negative-parity states turn out to be degenerate, while for  $N_f = 4n + 2$  the pseudoscalar condensate is suppressed or vanishes completely. This is also consistent with our analysis in section 4.1: For  $N_f = 4n$  the microscopic theory always has a positive definite measure so that we can derive Banks-Casher-type relations for both the scalar and the pseudoscalar diquark condensate, showing that their magnitudes are equal. For  $N_f = 4n + 2$  the pseudoscalar diquark source leads to an indefinite measure in the microscopic theory, and therefore we cannot conclude anything about the relative magnitude of the two condensates.

## 5.2 Effective theory L at low density

At low density the effective theory derived in [19, 20] uses as degrees of freedom the NG modes corresponding to the symmetry-breaking pattern (5.1) at zero density. We briefly review and extend the relevant results here. In the chiral limit, the leading-order effective Lagrangian is given by

$$\mathcal{L}_{\text{eff}}^{\mathbf{L}} = \frac{F^2}{2} \text{tr}(\nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger) - \Phi_{\mathbf{L}} \text{Re tr}(\bar{J} \Sigma) \quad (5.4)$$

with

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - \mu \delta_{\nu 0} (B \Sigma + \Sigma B), \quad (5.5)$$

$$\nabla_\nu \Sigma^\dagger = \partial_\nu \Sigma^\dagger + \mu \delta_{\nu 0} (\Sigma^\dagger B + B \Sigma^\dagger), \quad (5.6)$$

$$B = \begin{pmatrix} \mathbb{1}_{N_f} & 0 \\ 0 & -\mathbb{1}_{N_f} \end{pmatrix}, \quad \bar{J} = \begin{pmatrix} J_L & 0 \\ 0 & -J_R^\dagger \end{pmatrix}. \quad (5.7)$$

The field  $\Sigma$  parametrizes the coset space  $\text{SU}(2N_f)/\text{Sp}(2N_f)$ ,

$$\Sigma = U \Sigma_d U^T \quad \text{with} \quad \Sigma_d = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad U = \exp\left(\frac{i\pi^a T^a}{2F}\right), \quad (5.8)$$

where the  $T^a$  ( $a = 1, \dots, N_f(2N_f - 1) - 1$ ) are the generators of  $\text{SU}(2N_f)/\text{Sp}(2N_f)$ . They are Hermitian and satisfy  $\text{tr}(T^a) = 0$  and  $\text{tr}(T^a T^b) = \delta_{ab}$ . A sum over the repeated index  $a$  in (5.8) is understood.

There are four minor differences with respect to [20]. First, only the case of  $J_R = -J_L$  was considered there. Second, our  $\Phi_{\mathbf{L}}$  corresponds to their  $G$ . Third, the analysis in [20] also includes a nonzero quark mass  $m$ . We only consider  $m = 0$ , in which case the chiral condensate immediately disappears as  $\mu$  is switched on, while the diquark condensate immediately assumes its full value, see section 12 of [20]. Fourth, to be consistent with the rest of the current paper, our convention for the diquark source and consequently for  $\Sigma_d$  differs from [20]. This follows from requiring parity invariance of the microscopic Lagrangian (which implies  $J_L \leftrightarrow -J_R$  under parity), of the diquark condensate (which

determines the form of  $\Sigma_d$ ), and of the effective theory (which determines the form of  $\bar{J}$  and the second term in (5.4)). As a consistency check, we note that the minimum of the action is obtained for  $\Sigma = \Sigma_d$  and  $J_R = -J_L = jI$  with  $j$  real and positive.

In (5.4) there are two low-energy constants.  $F$  is the common decay constant of all NG modes, and the positive parameter  $\Phi_{\mathbf{L}}$  is the magnitude of the diquark condensate per flavor and handedness in the absence of sources and at  $\mu = 0$ ,

$$\Phi_{\mathbf{L}} = \frac{1}{N_f} \left| \langle \psi_i^T C \tau_2 I \psi_i \rangle \right|_{\bar{J}=0, \mu=0} \quad (i = L, R). \quad (5.9)$$

It is important to note that  $F$  and  $\Phi_{\mathbf{L}}$  do *not* depend on  $\mu$ .

For  $J_R = -J_L = jI$  the masses of the NG modes have been computed in [20, eq. (101)], and in the chiral limit we have  $\varphi = \alpha = \pi/2$  and  $m_\pi = \sqrt{j\Phi}/F$  in these expressions. Hence there are two types of NG modes,

$$\text{type 1:} \quad \text{mass} = \sqrt{j\Phi_{\mathbf{L}}/F^2} \quad (N_f^2 - N_f - 1 \text{ modes}), \quad (5.10a)$$

$$\text{type 2:} \quad \text{mass} = \sqrt{j\Phi_{\mathbf{L}}/F^2 + (2\mu)^2} \quad (N_f^2 \text{ modes}). \quad (5.10b)$$

While the type-1 modes are massless in the  $j \rightarrow 0$  limit, (5.10b) shows that  $\mu$  is an explicit symmetry-breaking parameter that makes the type-2 modes massive even for  $j = 0$ .

Let us comment on the source  $J_L = J_R$  for the pseudoscalar diquark condensate.<sup>27</sup> For  $N_f = 4n$  we define  $u = \text{diag}(i\mathbb{1}_{4n}, \mathbb{1}_{4n})$ . Since  $u \in \text{SU}(8n)$  we can redefine  $U \rightarrow uU$  in (5.8). The measure is not changed by this transformation, but in the term  $\bar{J}\Sigma$  in (5.4) the sign of  $J_L$  in (5.7) is flipped. This means that the partition functions (and hence the energies) for  $J_L = -J_R$  and for  $J_L = J_R$  are exactly equal. This is consistent with the observation that the instanton vertex does not prefer one condensate over the other for  $N_f = 4n$ , see the discussion at the end of section 5.1. For  $N_f = 2$  we find that a pseudoscalar diquark source term drops out from (5.4) since  $\text{Re tr}(\bar{J}\Sigma) = \text{Re tr}[\text{diag}(I, I)U \text{diag}(I, -I)U^T] = 0$  for  $U \in \text{SU}(4)$ .<sup>28</sup> Hence the pseudoscalar diquark condensate is zero (in this order of the low-energy expansion), which is again consistent with the instanton-based argument in section 5.1. For  $N_f = 4n + 2$  with  $n \geq 1$  the situation is more complicated. While we currently cannot make any definite analytical statement, numerical experimentation indicates that for  $J_R = J_L = jI$  the minimum of the energy is larger than for  $J_R = -J_L = jI$ . If true, this would mean that by changing the diquark source we can generate a nonzero pseudoscalar diquark condensate whose magnitude is smaller than that of the scalar condensate.

### 5.3 Effective theory H at high density

For technical reasons we now proceed to the regime of very high density. The effective chiral Lagrangian including mass term for this case was derived in [15]. Here, we are interested in diquark sources and therefore set the mass term to zero.

<sup>27</sup>This is equivalent to  $J_L = -J_R$  with  $\theta = N_f\pi/2$ , see section 2.1, i.e., instead of studying the  $J$ -dependence we could equivalently study the  $\theta$ -dependence. This statement applies at any density.

<sup>28</sup>This can be shown using an explicit parameterization of the coset space  $\text{SU}(4)/\text{Sp}(4)$ , see, e.g., [71].



We first consider the case of  $N_f \geq 4$ . The symmetry-breaking pattern is given in (5.3). By forming linear combinations of the generators of  $U(1)_B$  and  $U(1)_A$  we can switch from  $U(1)_B \times U(1)_A$  to  $U(1)_L \times U(1)_R$ . We parametrize the NG modes as

$$\Sigma_i = U_i I U_i^T \quad \text{with} \quad U_i = \exp\left(\frac{i\pi_i^a T^a}{2\tilde{f}}\right) \quad (i = L, R), \quad (5.11)$$

$$L = \exp\left(\frac{i\pi_L^0}{\sqrt{N_f}\tilde{f}_0}\right), \quad R = \exp\left(\frac{i\pi_R^0}{\sqrt{N_f}\tilde{f}_0}\right). \quad (5.12)$$

The parameterization of the  $\Sigma_i$  is similar to (5.8), except that the  $T^a$  ( $a = 1, \dots, N_f(N_f - 1)/2 - 1$ ) are now the Hermitian generators of  $SU(N_f)/Sp(N_f)$ , again satisfying  $\text{tr}(T^a) = 0$  and  $\text{tr}(T^a T^b) = \delta_{ab}$ .<sup>29</sup> Using  $U_i I U_i^T = U_i^2 I$  [20] we can also write  $\Sigma_i = U_i^2 I$ .

The quarks transform under  $SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R$  as

$$\psi_L \rightarrow e^{i\alpha_L} g_L \psi_L, \quad \psi_R \rightarrow e^{i\alpha_R} g_R \psi_R, \quad (5.13)$$

where  $g_i \in SU(N_f)_i$  and  $e^{i\alpha_i} \in U(1)_i$  ( $i = L, R$ ). The NG modes therefore transform as

$$\Sigma_i \rightarrow g_i \Sigma_i g_i^T \quad (i = L, R), \quad L \rightarrow L e^{2i\alpha_L}, \quad R \rightarrow R e^{2i\alpha_R}. \quad (5.14)$$

The transformation properties of  $J_L$  and  $J_R$  are determined by requiring that the Lagrangian (2.3) be invariant under the flavor symmetries. This implies

$$J_L \rightarrow g_L^* J_L g_L^\dagger e^{-2i\alpha_L}, \quad J_R \rightarrow g_R^* J_R g_R^\dagger e^{-2i\alpha_R}. \quad (5.15)$$

Therefore the invariant real combination linear in  $J_L$  and  $J_R$  is uniquely determined to be

$$\text{Re} [L \text{tr}(J_L \Sigma_L) - R \text{tr}(J_R \Sigma_R)]. \quad (5.16)$$

As for theory **L**, we required parity invariance of the microscopic theory (implying  $J_L \leftrightarrow -J_R$ ), of the diquark condensate (implying  $\Sigma_L \leftrightarrow \Sigma_R$  and  $L \leftrightarrow R$ ), and of the effective theory (leading to the relative factor of  $-1$  in (5.16)). The leading-order effective Lagrangian in the presence of diquark sources and in the chiral limit is thus given by

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{\mathbf{H}} = & \left[ \frac{N_f \tilde{f}_0^2}{2} (|\partial_0 L|^2 + \tilde{v}_0^2 |\partial_i L|^2) + \frac{\tilde{f}^2}{2} \text{tr} (|\partial_0 \Sigma_L|^2 + \tilde{v}^2 |\partial_i \Sigma_L|^2) + (L \leftrightarrow R) \right] \\ & - \Phi_{\mathbf{H}} \text{Re} \text{tr}(J_L L \Sigma_L - J_R R \Sigma_R) - \frac{2\tilde{f}_0^2}{N_f} m_{\text{inst}}^2 \text{Re} (L^\dagger R)^{N_f/2}, \end{aligned} \quad (5.17)$$

where  $\tilde{f}_0, \tilde{f}$  and  $\tilde{v}_0, \tilde{v}$  are low-energy constants that correspond to the decay constants and velocities of the NG modes, respectively. The latter are generally different from unity (i.e., the speed of light) since Lorentz invariance is lost at  $\mu \neq 0$ . The minus sign in front of the positive low-energy constant  $\Phi_{\mathbf{H}}$  is chosen so that the minimum of the action is obtained for  $L = R = 1$ ,  $\Sigma_L = \Sigma_R = I$ , and  $J_R = -J_L = jI$  with  $j$  real and positive. Note that all

<sup>29</sup>The  $T^a$  are related to the  $X^a$  in [15] by  $T^a = X^a/\sqrt{N_f}$ .



low-energy “constants” in  $\mathcal{L}_{\text{eff}}^{\mathbf{H}}$  depend on  $\mu$ . Their relation to physical observables will be discussed in section 5.6.

In (5.17) we have also included a term that corresponds to the single-instanton contribution to the  $\eta'$  mass, parametrized by  $m_{\text{inst}}$ . This term is symmetric under the anomaly-free subgroup  $(\mathbb{Z}_{2N_f})_A$  of  $U(1)_A$ . From the symmetry point of view all terms of the form  $\text{Re}(L^\dagger R)^{nN_f/2}$  ( $n \geq 1$ ) are allowed and contribute to the  $\eta'$  mass. Microscopically, these terms correspond to  $n$ -instanton vertices. At sufficiently large  $\mu$ , the instanton ensemble can be regarded as a dilute gas [66–68] which does not form instanton molecules [72]. The diluteness of the instanton gas is parametrized by the dimensionless quantity (proportional to the instanton density)  $a \propto (\Lambda_{\text{QCD}}/\mu)^{b(N_f)} \ll 1$  with  $b(N_f) = (22 - 2N_f)/3$ . Since the probability  $\sim a^n$  that  $n$  instantons ( $n \geq 2$ ) are at the same point is highly suppressed we can neglect the multi-instanton vertices and only keep the one-instanton contribution. Note that  $m_{\text{inst}}$  is a decreasing function of  $\mu$ , with  $m_{\text{inst}} \rightarrow 0$  for  $\mu \rightarrow \infty$  [66–68].

It is convenient to combine the  $U(1)_i$  field with  $SU(N_f)_i/\text{Sp}(N_f)_i$  by defining

$$\tilde{\Sigma}_L = L\Sigma_L = \exp\left(\frac{i\pi_L^A T^A}{\tilde{f}_A}\right) I \quad \text{and} \quad (L \leftrightarrow R), \quad (5.18)$$

where the  $T^A$  ( $A = 0, \dots, N_f(N_f - 1)/2 - 1$ ) are now the generators of  $U(N_f)/\text{Sp}(N_f)$  with  $T^0 = \mathbb{1}/\sqrt{N_f}$  so that  $\text{tr}(T^A T^B) = \delta_{AB}$ .<sup>30</sup> We also defined  $\tilde{f}_A = \tilde{f}$  for  $A \geq 1$ . To second order in the  $\pi$ -fields we have

$$\text{Re } \tilde{\Sigma}_i = \left(1 - \frac{\pi_i^A \pi_i^B T^A T^B}{2\tilde{f}_A \tilde{f}_B}\right) I. \quad (5.19)$$

Assuming  $J_R = -J_L = jI$  with  $j$  real and positive, this yields a Gell-Mann–Oakes–Renner (GOR) type mass formula for the  $\pi_i^A$ . As in theory **L** there are two types of NG modes,

$$\text{type 1:} \quad m_A = \sqrt{j\Phi_{\mathbf{H}}/\tilde{f}_A^2} \quad (N_f^2 - N_f - 1 \text{ modes}), \quad (5.20a)$$

$$\text{type 2:} \quad m_{\eta'} = \sqrt{j\Phi_{\mathbf{H}}/\tilde{f}_0^2 + m_{\text{inst}}^2} \quad (1 \text{ mode}). \quad (5.20b)$$

Note that there are two type-1 modes for each  $A \geq 1$ , but only a single one for  $A = 0$ . Note also the similarity with (5.10). Now  $m_{\text{inst}}$  plays the role of the symmetry-breaking parameter which makes the type-2 mode massive as  $\mu$  is lowered.

Let us now consider the case of  $N_f = 2$ , in which  $\Sigma_L$  and  $\Sigma_R$  are absent because  $SU(2) \sim \text{Sp}(2)$ . Thus the effective Lagrangian contains only the fields  $L$  and  $R$ . Since any  $2 \times 2$  antisymmetric matrix is proportional to  $I$  we can write  $J_L = j_L I$  and  $J_R = j_R I$  ( $j_L, j_R \in \mathbb{C}$ ) without loss of generality. These terms transform as

$$j_L \rightarrow j_L e^{-2i\alpha_L}, \quad j_R \rightarrow j_R e^{-2i\alpha_R}, \quad (5.21)$$

which follows from (5.15) and  $g_i^* I g_i^\dagger = (\det g_i^*) I = I$  ( $i = L, R$ ). Therefore the invariant real combination linear in  $j_R$  and  $j_L$  is given by

$$\text{Re}(j_L L - j_R R). \quad (5.22)$$

<sup>30</sup>We use the convention that uppercase indices corresponding to  $U(N_f)/\text{Sp}(N_f)$  start at zero, while lowercase indices corresponding to  $SU(N_f)/\text{Sp}(N_f)$  start at 1.

Using the same parameterization of  $L$  and  $R$  as in (5.12), the effective Lagrangian for  $N_f = 2$  reads

$$\mathcal{L}_{\text{eff}}^{\mathbf{H}} = \tilde{f}_0^2 [|\partial_0 L|^2 + \tilde{v}_0^2 |\partial_i L|^2 + (L \leftrightarrow R)] + 2\Phi_{\mathbf{H}} \text{Re}(j_L L - j_R R) - \tilde{f}_0^2 m_{\text{inst}}^2 \text{Re}(L^\dagger R), \quad (5.23)$$

where the plus sign in front of  $\Phi_{\mathbf{H}}$  and the factor of  $-1$  between the two terms following it have been chosen so that the minimum of the action is obtained for  $L = R = 1$  and  $j_R = -j_L = j$  with  $j$  real and positive. The GOR-type relation for this case is identical to (5.20) with  $A = 0$ .

We again comment on the case of  $J_L = J_R$ . Note first that for  $m_{\text{inst}} = 0$  (i.e., at asymptotically high density) the fields  $L$  and  $R$  in (5.17) or (5.23) can rotate independently, and hence the left and right diquark condensates in the ground state can be rotated separately by varying the directions of  $J_L$  and  $J_R$ . This is no longer true when  $m_{\text{inst}} \neq 0$ . Since the anomaly term favors  $L = R$  energetically, the fields  $L$  and  $R$  can no longer rotate independently. Now let us again discuss the various cases of  $N_f$ . For  $N_f = 4n$  we can redefine  $U_L \rightarrow u U_L$  in (5.11) with  $u = \text{diag}(i\mathbb{1}_{4n}) \in \text{SU}(4n)$ , which flips the sign of  $\Sigma_L$  and thus absorbs a sign flip of  $J_L$  in (5.17). So again the energies for  $J_L = -J_R$  and  $J_L = J_R$  are equal, in agreement with the instanton-based argument in section 5.1. For  $N_f = 2$  and  $j_R = -j_L > 0$  the diquark source term and the anomaly term in (5.23) are minimized simultaneously at  $L = R = 1$ , and thus the ground state is not changed by the anomaly term. However, for  $j_R = j_L > 0$  there is a competition between these two terms, and they cannot be minimized simultaneously. Therefore the pseudoscalar diquark condensate can be realized only if the diquark sources are strong enough to overcome the penalty due to the anomaly term. For  $N_f = 4n + 2$  with  $n \geq 1$  the situation is similar but a bit more complicated. For  $J_R = -J_L = jI$  both the diquark source term and the anomaly term in (5.17) can be minimized simultaneously. For  $J_R = J_L = jI$  we suspect, although we currently cannot prove it analytically, that this cannot be done. A more quantitative study is required to determine the magnitude of the pseudoscalar diquark condensate in this case.

#### 5.4 Effective theory I at intermediate density

At intermediate density, the coupling constant is not small enough to treat instantons as a dilute screened gas, and hence the  $\text{U}(1)_A$  anomaly can no longer be treated as a small perturbation. In other words,  $m_{\text{inst}}$  increases as  $\mu$  is lowered so that the  $\eta'$  mass in (5.20b) is not necessarily small, implying that the  $\eta'$  should be integrated out from the effective theory. Technically, this means that  $L$  and  $R$  should be replaced by a single  $\text{U}(1)$  field  $V$ , i.e., the effective Lagrangian (5.17) for  $N_f \geq 4$  changes to

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{\mathbf{I}} = & N_f f_0^2 [|\partial_0 V|^2 + v_0^2 |\partial_i V|^2] + \frac{f^2}{2} \text{tr} [|\partial_0 \Sigma_L|^2 + v^2 |\partial_i \Sigma_L|^2 + (L \leftrightarrow R)] \\ & - \Phi_{\mathbf{I}} \text{Re} [V \text{tr}(J_L \Sigma_L - J_R \Sigma_R)] \end{aligned} \quad (5.24)$$

corresponding to the symmetry-breaking pattern (5.2). We have renamed the low-energy “constants” to distinguish them from those of theory  $\mathbf{H}$ . Note that they again depend on

$\mu$ . In section 5.6 we will discuss how they are related to the low-energy constants of theory **L** and **H** at low and high density, respectively.

The NG modes  $\Sigma_{L/R}$  are parametrized as in (5.11) with  $\tilde{f}$  replaced by  $f$ , while

$$V = \exp \left( \frac{i\pi_V^0}{\sqrt{2N_f}f_0} \right). \quad (5.25)$$

To second order in the  $\pi$ -fields we now have

$$\text{Re}(\Sigma_i V) = \left( 1 - \frac{(\pi_V^0)^2}{4N_f f_0^2} - \frac{\pi_V^0 \pi_i^a T^a}{\sqrt{2N_f} f_0 f} - \frac{\pi_i^a \pi_i^b T^a T^b}{2f^2} \right) I. \quad (5.26)$$

Assuming again  $J_R = -J_L = jI$  with  $j$  real and positive, we obtain a GOR-type mass formula analogous to (5.20a),

$$m_A = \sqrt{j\Phi_{\mathbf{I}}/f_A^2} \quad (N_f^2 - N_f - 1 \text{ modes}), \quad (5.27)$$

where  $f_A = f$  for  $A \geq 1$ . Note that we have only type-1 modes in theory **I**.

For  $N_f = 2$  the effective Lagrangian changes to

$$\mathcal{L}_{\text{eff}}^{\mathbf{I}} = 2f_0^2 [|\partial_0 V|^2 + v_0^2 |\partial_i V|^2] + 2\Phi_{\mathbf{I}} \text{Re}[(j_L - j_R)V], \quad (5.28)$$

and the GOR-type relation for the single NG mode is  $m_0 = \sqrt{j\Phi_{\mathbf{I}}/f_0^2}$  as in (5.27).

Finally, we again consider the case of  $J_L = J_R$ . For  $N_f = 4n$  the argument and the conclusion are exactly the same as in theory **H**. For  $N_f = 2$  the pseudoscalar diquark source term in (5.28) drops out trivially since  $j_L = j_R$ , and hence the pseudoscalar diquark condensate is zero (in this order of the low-energy expansion) as in theory **L**. For  $N_f = 4n+2$  we suspect, although we currently cannot prove it analytically, that for  $J_R = J_L = jI$  the minimum of the ground-state energy is larger than for  $J_R = -J_L = jI$ . If true, we are led to the same conclusion as in theory **L**.

## 5.5 Domains of validity

In the following discussion we assume that the diquark sources are infinitesimal. In general, there are two conditions for an effective theory formulated in terms of NG modes to be applicable: (i) the masses of all NG modes must be much smaller than the mass scale  $m_\ell$  of the lightest non-NG particle, and (ii) the typical scale  $p$  of observables computed within the effective theory must also be much smaller than  $m_\ell$ . Of course,  $m_\ell$  itself must be nonzero. To figure out the domains of validity of the three effective theories at nonzero density we must determine the mass scale  $m_\ell$  of each theory, which generically is a function of  $\mu$ .

For the effective theory **L** we have  $m_\ell(\mathbf{L}) \sim \Lambda$ , where  $\Lambda$  is the mass of the lightest non-NG particle at zero density. As  $\mu$  is increased from zero, some of the NG modes of **L** acquire a mass proportional to  $\mu$ . The effective theory **I** is obtained from **L** by integrating out these modes so that for **I** at low density we have  $m_\ell(\mathbf{I}) \sim \mu$ .

The situation at high density is somewhat more complicated. At asymptotically high density the  $\eta'$  is massless and  $\Delta \ll \mu$ . For the effective theory **H** we have  $m_\ell(\mathbf{H}) \sim \Delta$ .

effective theory	condition (i)	condition (ii)
<b>L</b>	$\mu \ll \Lambda$	$p \ll \Lambda$
<b>I</b> (scenario 1)	—	$p \ll \mu$ for $\mu \lesssim \Lambda$ $p \ll \Delta$ for $\mu_{\text{BCS}} \lesssim \mu \lesssim \mu_c$ $p \ll m_{\eta'}$ for $\mu \gtrsim \mu_c$
<b>I</b> (scenario 2)	—	$p \ll \mu$ for $\mu \lesssim \Lambda$ $p \ll m_{\eta'}$ for $\mu \gtrsim \mu_{\text{BCS}}$
<b>H</b>	$m_{\eta'} \ll \Delta$	$p \ll \Delta$

**Table 1.** Domains of validity of the three effective theories **L**, **I**, and **H**, see text for details.

There are two ways to see this. First,  $\Delta$  plays the role of a constituent quark mass so that the lightest non-NG particles (color singlet diquarks and mesons) weigh about  $2\Delta$  [22]. Second, the higher-order vertices in the effective Lagrangian are suppressed by  $1/\Delta$ , while loop integrals are suppressed by  $1/\tilde{f}_A$  [73]. Since  $\tilde{f}_A \sim \mu$  [74] and  $\Delta \ll \mu$  the cutoff is  $\Delta$ . Let us now lower the density so that the  $\eta'$  becomes massive, but let  $\mu$  be large enough so that we still have  $m_{\eta'} \ll \Delta \ll \mu$  [66–68]. The effective theory **I** is now obtained from **H** by integrating out the  $\eta'$  so that for **I** in this regime we have  $m_\ell(\mathbf{I}) \sim m_{\eta'}$ . As the density is lowered further there are two possible scenarios:

1. There could be a “critical” chemical potential  $\mu_c$  at which  $m_{\eta'} = \Delta$  and below which  $m_{\eta'} > \Delta$ . In that case  $m_\ell(\mathbf{I}) \sim \Delta$  for  $\mu_{\text{BCS}} \lesssim \mu \lesssim \mu_c$ , where  $\mu_{\text{BCS}}$  is the chemical potential above which we are in the BCS regime.<sup>31</sup>
2. We could have  $m_{\eta'} < \Delta$  for all  $\mu \gtrsim \mu_{\text{BCS}}$ . Then  $m_\ell(\mathbf{I}) \sim m_{\eta'}$  for all  $\mu \gtrsim \mu_{\text{BCS}}$ .

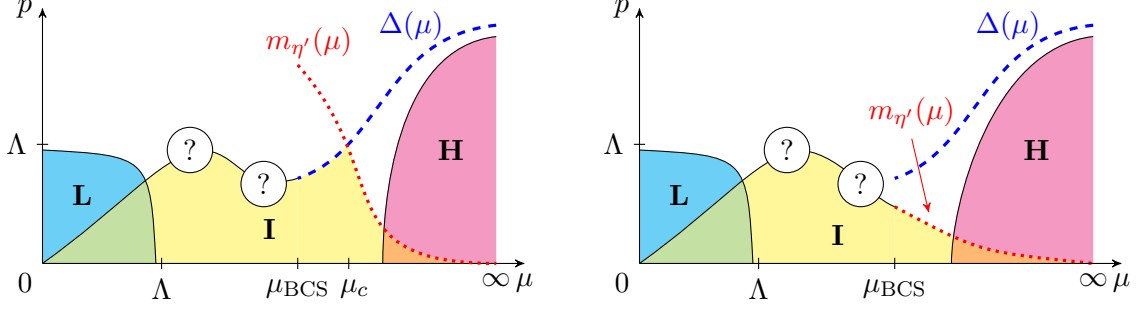
Since we only know the functions  $\Delta(\mu)$  and  $m_{\eta'}(\mu)$  at asymptotically large  $\mu$  we do not know which of these two scenarios is correct.<sup>32</sup> This can only be decided by a full dynamical calculation, e.g., in lattice QCD. However, the two scenarios can be combined in the statement  $m_\ell(\mathbf{I}) \sim \min(\Delta(\mu), m_{\eta'}(\mu))$  for  $\mu \gtrsim \mu_{\text{BCS}}$ .<sup>33</sup>

Our discussion is summarized in table 1 and figure 4. Three comments are in order. First, condition (i) is always satisfied for the effective theory **I** since the  $N_f(N_f - 1) - 1$  NG modes shown in figure 3 are always massless. In other words, **I** is applicable at any  $\mu$  as long as the scale of the observable is sufficiently small. Second, as the “intermediate” density is increased from “low” to “high”, the mass scale of the lightest non-NG mode changes from  $\mu$  to  $m_{\eta'}$ . Finding the precise  $\mu$ -dependence of  $m_\ell$  in the intermediate region again requires a full dynamical calculation. All we currently know are the two limits  $\mu$

<sup>31</sup>Although there is no phase transition between the BEC and BCS regimes, we can define  $\mu_{\text{BCS}}$  as the chemical potential above which the minimum of the dispersion relation of the fermionic quasiparticles changes from  $p = 0$  to  $p \neq 0$ , with  $p$  being the momentum [75].

<sup>32</sup>In these asymptotic functions we always have  $m_{\eta'} < \Delta$ , but this does not tell us anything about the regime where  $\mu$  is not asymptotically large.

<sup>33</sup>In the preceding arguments we have completely ignored gluons, even though they are lighter than the  $\eta'$  at sufficiently large  $\mu$  [68], since their interaction with NG modes is assumed to be negligibly small, see the discussion in [15].



**Figure 4.** Domains of validity of the three effective theories **L**, **I**, and **H** in scenario 1 (left) and scenario 2 (right), see text for details.

and  $m_{\eta'}$ . Third, in the overlap regions of the different effective theories (i.e., the green and orange areas in figure 4) one has a choice of which theory to use, but this choice depends on the observable. (This is similar to the choice between SU(2) and SU(3) flavor chiral perturbation theory in QCD as a function of the strange quark mass [76, 77].) For example, in the regime  $\mu \ll \Lambda$  one could also use the effective theory **I**, but this would only work for observables with  $p \ll \mu$ , whereas the effective theory **L** could be used for observables with  $p \ll \Lambda$ . Similarly, the effective theory **I** could also be used in the regime  $m_{\eta'} \ll \Delta$ . This would only work for observables with  $p \ll m_{\eta'}$ , whereas the effective theory **H** could be used for observables with  $p \ll \Delta$ . While at first sight it may seem that one should always work with the effective theory that allows for a larger range of observables, it may be technically simpler to work with the effective theory **I** if one is only interested in an observable for which this effective theory is valid.

## 5.6 Matching of the low-energy constants

Let us now comment on the relation between low-energy constants and physical observables, and on the matching of the low-energy constants between the different effective theories. Note that there are also low-energy constants corresponding to higher-orders in the effective Lagrangians, which we have not shown explicitly.

Let us start with theory **I**, because this is simplest as the diquark source is the only symmetry-breaking perturbation. In that case  $f_0$  and  $f$  are equal to the physical decay constants at a given  $\mu$  in the limit  $J \rightarrow 0$ , and

$$\Phi_{\mathbf{I}} = \frac{1}{N_f} |\langle \psi_i^T C \tau_2 I \psi_i \rangle|_{J=0} \quad (i = L, R). \quad (5.29)$$

For nonzero  $J$  there will be corrections (similar to the chiral corrections in chiral perturbation theory) due to which the low-energy constants will deviate from the physical quantities. Next we consider theory **L**. It has two symmetry-breaking perturbations,  $\mu$  and  $J$ . The low-energy constants do not depend on these external parameters. Rather, they are equal to the decay constants and the diquark condensate at  $\mu = J = 0$ , see (5.9). Now let us recall that the theories **L** and **I** have overlapping domains of validity at low density. All physical quantities should be independent of which effective theory we use. As long

as we work at any *finite* order of the low-energy expansion in theory **L**, the results thus obtained could be different from those of theory **I**. However, the discrepancy will disappear if we sum up the contributions of the heavier NG modes (with mass  $\sim \mu$ ) in theory **L** to all orders.<sup>34</sup> We expect, for any fixed  $\mu \ll \Lambda$ , the relation

$$|\langle \psi \psi \rangle|_{J=0} = \Phi_{\mathbf{I}} = \Phi_{\mathbf{L}} + (\text{Corrections due to the propagation of type-2 modes}), \quad (5.30)$$

and likewise for  $F$ ,  $f_0$ , and  $f$ . The relations between the low-energy constants of theory **L** and **I** can be made more precise by explicitly integrating out the type-2 modes of theory **L**. In the course of this procedure, the (initially  $\mu$ -independent) low-energy constants acquire a  $\mu$ -dependence in much the same way as the low-energy constants of SU(2) chiral perturbation theory acquire a dependence on the strange quark mass  $m_s$  when kaons are integrated out of SU(3) chiral perturbation theory [76, 77, 79, 80]. According to these calculations, the corrections to the low-energy constants at  $O(p^2)$  ( $F$  and  $B$  in the standard notation) due to the integrating-out of kaons become arbitrarily small when  $m_s$  gets small. If we assume that this finding persists in our present context, we expect

$$\lim_{\mu \rightarrow 0} f_0, f = F \quad \text{and} \quad \lim_{\mu \rightarrow 0} \Phi_{\mathbf{I}} = \Phi_{\mathbf{L}}. \quad (5.31)$$

However, we do not expect this smooth matching to extend to the low-energy constants of higher orders, because it is known that in SU(2) chiral perturbation theory they receive corrections of the form  $(1/m_s)^n$  with  $n > 0$  and thus blow up as  $m_s \rightarrow 0$ . Hence they cannot reduce to the low-energy constants of SU(3) chiral perturbation theory [76, 77, 79, 80].

The discussion for theory **H** and its matching with theory **I** proceeds analogously. In theory **H**, the role of  $\mu$  in theory **L** is now played by  $m_{\text{inst}}$ , which in turn is a function of  $\mu$ . For sufficiently high density the domains of validity of theory **H** and **I** overlap, and for any fixed  $\mu$  (provided that  $m_{\text{inst}} \ll \Delta$ ) we expect the relation

$$|\langle \psi \psi \rangle|_{J=0} = \Phi_{\mathbf{I}} = \Phi_{\mathbf{H}} + (\text{Corrections due to the propagation of } \eta') \quad (5.32)$$

to hold (and likewise for  $f_0$ ,  $\tilde{f}_0$  and  $f$ ,  $\tilde{f}$ ). Based on our discussion at low density we now expect the matching (at any fixed  $\mu$ )

$$\lim_{m_{\text{inst}} \rightarrow 0} f_0, f = \tilde{f}_0, \tilde{f} \quad \text{and} \quad \lim_{m_{\text{inst}} \rightarrow 0} \Phi_{\mathbf{I}} = \Phi_{\mathbf{H}}. \quad (5.33)$$

However, there is a subtlety specific to high density. For  $\mu \rightarrow \infty$  the low-energy “constants” are actually infinite since  $f_0, f \sim \mu$  [74] and  $\Phi \sim \mu^2 \Delta / g$  [81]. Therefore  $\tilde{f}_0$ ,  $\tilde{f}$ , and  $\Phi_{\mathbf{H}}$  cannot be defined as constants at  $\mu = \infty$ . Accordingly, theory **H** cannot be defined at  $\mu = \infty$  in the same way as theory **L** could be defined at  $\mu = 0$ .<sup>35</sup> That is why we defined

<sup>34</sup>This is similar to what is encountered in weak-coupling perturbation theory. In this case observables (such as the cross section) depend on the renormalization scale at any finite order of perturbation theory, but this dependence decreases as we go to higher orders [78].

<sup>35</sup>This is not meant as a negative statement since the asymptotic behavior of the low-energy constants in terms of  $\mu$  is known. Similar situations occur when considering the large- $N_c$  limit or scattering processes in the limit of high energies.

theory **H** and its low-energy “constants” at fixed (and finite)  $\mu$ . This implies that  $m_{\text{inst}}$  is also fixed and cannot be considered as an independent symmetry-breaking parameter anymore. However, we can start from theory **H** at a given  $\mu$  and formally integrate out the  $\eta'$  to obtain theory **I** at the same value of  $\mu$ . The low-energy “constants” of **I** thus acquire a dependence on  $m_{\text{inst}}$ , and we can now formally send  $m_{\text{inst}} \rightarrow 0$ , still at the same fixed  $\mu$ . This is how (5.33) should be understood.

At the end of this section, let us comment on possible extensions. We could have performed a more comprehensive analysis with nonzero diquark sources, which would give a small mass proportional to  $\sqrt{j}$  to the NG modes. However, in this paper we are only interested in the limit  $j \rightarrow 0$ , and therefore we have not performed such an analysis.<sup>36</sup> In principle one could also add explicit quark masses. This makes the analysis even more complicated since the coset space one should use to construct the effective theory now depends not only on  $\mu$  but also on the quark masses. From the arguments presented in this section it should be clear how to proceed, but we do not pursue this issue further.

## 6 Smilga-Stern-type relations ( $\beta = 1$ )

In [2] Smilga and Stern computed the slope of the density of Dirac eigenvalues at the origin in the QCD vacuum ( $\beta = 2$ ) using effective-theory techniques. Their result was confirmed by partially quenched chiral perturbation theory for degenerate [82] and nondegenerate [83] masses. Later it was generalized to theories with  $\beta = 1$  and 4 at  $\mu = 0$  [84]. In this section we adapt the method of [2] to the singular values at  $\mu \neq 0$ , i.e., we compute the slope of the singular value density of the Dirac operator in two-color QCD at nonzero  $\mu$  ( $\beta = 1$ ), using the effective theories constructed in section 5. We will obtain three different results at infinite, intermediate, and zero density, respectively. In section 6.4 we will discuss the relation between these results as a function of  $\mu$ . Throughout this section we work in the chiral limit for simplicity.

### 6.1 Infinite density

For technical reasons we now start at infinite density so that we can set  $m_{\text{inst}} = 0$  in (5.17). As in earlier sections we set  $J_R = -J_L = J$ , where  $J$  is an antisymmetric  $N_f \times N_f$  matrix that has  $N_f(N_f - 1)/2$  independent components. We can decompose  $J$  as

$$J = I \sum_A j_A t^A = jI + I \sum_a j_a t^a, \quad (6.1)$$

where the  $t^A$  are the generators of  $U(N_f)/\text{Sp}(N_f)$  and the  $j_A$  are real parameters with  $j_0 = j\sqrt{N_f}$ . Such a decomposition is possible since the dimension of  $U(N_f)/\text{Sp}(N_f)$  is  $N_f(N_f - 1)/2$  and thus equal to the number of degrees of freedom of  $J$ , and since  $It^A$  is antisymmetric for all  $A$  (which follows from  $t^A I = I(t^A)^T$  [20]). As before, the sum over  $A$  starts at 0, while the sum over  $a$  starts at 1. The  $t^A$  are identical to the  $T^A$  defined below (5.18), but we denote them by a different symbol (in agreement with the notation

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<sup>36</sup>But see section 6.4 in which we are forced to consider the case of small but nonzero  $j$  to understand an apparent discontinuity of a very particular observable.



of [84]) since they are used in a different context: The  $T^A$  are used to parametrize the NG modes of the effective theory living in  $U(N_f)_L/\text{Sp}(N_f)_L \times U(N_f)_R/\text{Sp}(N_f)_R$ , while the  $t^A$  are used to parametrize the source  $J$ , which exists already in the microscopic theory. The choice of  $U(N_f)/\text{Sp}(N_f)$  to parametrize  $J$  is natural since it is the space in which the diquark condensate aligns depending on the choice of  $J$ .

For  $N_f \geq 4$  we now consider the scalar susceptibility

$$K_{ab}(j) = \lim_{V_4 \rightarrow \infty} \frac{1}{V_4} \partial_{j_a} \partial_{j_b} \ln Z(J) \Big|_{\text{all } j_a=0}, \quad (6.2)$$

which we will calculate both from the microscopic theory (two-color QCD) and from the low-energy effective theory.

Let us start on the QCD side. Assuming that at high density there are no zero modes, (2.10) yields

$$Z(J) = \left\langle \det^{1/2}(D^\dagger D + J^\dagger J) \right\rangle_{\text{YM}} = \left\langle \prod_n \det^{1/2}(\xi_n^2 + J^\dagger J) \right\rangle_{\text{YM}}. \quad (6.3)$$

After a bit of algebra we obtain from (6.2)

$$K_{ab}(j) = \delta_{ab} \lim_{V_4 \rightarrow \infty} \frac{1}{V_4} \left\langle \sum_n \frac{\xi_n^2 - j^2}{(\xi_n^2 + j^2)^2} \right\rangle_j = \delta_{ab} \int_0^\infty d\xi \rho_{\text{sv}}(\xi; j) \frac{\xi^2 - j^2}{(\xi^2 + j^2)^2}, \quad (6.4)$$

where  $\rho_{\text{sv}}(\xi; j)$  is defined as in (4.3) but with nonzero  $j$ .

On the low-energy effective theory side we need to differentiate the log of the effective partition function

$$Z_{\text{eff}}^{\mathbf{H}} = \int \mathcal{D}\tilde{\Sigma}_L \mathcal{D}\tilde{\Sigma}_R \exp \left( - \int d^4x \mathcal{L}_{\text{eff}}^{\mathbf{H}} \right) \quad (6.5)$$

with respect to  $j_a$  and  $j_b$ . Using (5.17) with  $m_{\text{inst}} = 0$  and (5.19) we obtain

$$\begin{aligned} K_{ab}(j) &= \sum_{ABCD} \frac{\Phi_{\mathbf{H}}^2}{4\tilde{f}_A\tilde{f}_B\tilde{f}_C\tilde{f}_D} \text{tr}(t^a T^A T^B) \text{tr}(t^b T^C T^D) \\ &\quad \times \frac{1}{V_4} \left\langle \int d^4x d^4y (\pi_L^A \pi_L^B + \pi_R^A \pi_R^B)(x) (\pi_L^C \pi_L^D + \pi_R^C \pi_R^D)(y) \right\rangle_j \\ &= \sum_{AB} \frac{\Phi_{\mathbf{H}}^2}{8\tilde{f}_A^2 \tilde{f}_B^2} \text{tr}(t^a \{T^A, T^B\}) \text{tr}(t^b \{T^A, T^B\}) \\ &\quad \times \left\langle \int d^4x \left( \pi_L^A \pi_L^B(x) \pi_L^A \pi_L^B(0) + \pi_R^A \pi_R^B(x) \pi_R^A \pi_R^B(0) \right) \right\rangle_j^{\text{conn}}, \end{aligned} \quad (6.6)$$

where “conn” denotes the connected part of the correlation function. To obtain (6.6) we have done the contractions  $A = C$ ,  $B = D$  and  $A = D$ ,  $B = C$  and symmetrized in  $A, B$ . The contraction  $A = B$ ,  $C = D$  corresponds to disconnected diagrams and yields zero.<sup>37</sup>

<sup>37</sup>The contribution of this contraction is proportional to  $\sum_b \text{tr}(t^a T^b T^b) \propto \text{tr}(t^a) = 0$  since  $\sum_b T^b T^b$  is the difference of the quadratic Casimir operators of  $SU(N_f)$  and  $\text{Sp}(N_f)$  and hence proportional to  $\mathbb{1}$ .



The dependence of (6.6) on  $j$  is contained in the masses of the NG modes. Evaluating the connected part in one-loop approximation<sup>38</sup> we find that it diverges for  $j \rightarrow 0$ ,

$$\left\langle \int d^4x \pi_L^A \pi_L^B(x) \pi_L^A \pi_L^B(0) \right\rangle_j^{\text{conn}} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m_A^2)(p^2 + m_B^2)} \sim \frac{1}{16\pi^2} \ln \left( \frac{\tilde{\Lambda}}{j} \right), \quad (6.7)$$

where we have used (5.20) and  $\tilde{\Lambda}$  is the momentum cutoff of the integral, in which we have also absorbed  $\tilde{f}_{A,B}$  and  $\Phi_{\mathbf{H}}$ . Thus the dependence of the integral on  $A$  and  $B$  has disappeared, and (6.6) becomes

$$K_{ab}(j) \sim Q_{ab} \frac{\Phi_{\mathbf{H}}^2}{16\pi^2} \ln \left( \frac{\tilde{\Lambda}}{j} \right) \quad (6.8)$$

with

$$Q_{ab} = \sum_{AB} \frac{1}{4\tilde{f}_A^2 \tilde{f}_B^2} \text{tr}(t^a \{T^A, T^B\}) \text{tr}(t^b \{T^A, T^B\}), \quad (6.9)$$

where we have included a factor of 2 for the left- and right-handed NG modes in the loop. To evaluate  $Q_{ab}$  we consider three cases:

1. For  $A = B = 0$ , the contribution to  $Q_{ab}$  vanishes since  $\text{tr}(t^a) = 0$ ,

$$Q_{ab}^{(1)} = 0. \quad (6.10)$$

2. For  $A = 0$ ,  $B \neq 0$  and  $A \neq 0$ ,  $B = 0$ , the contribution to  $Q_{ab}$  reads

$$Q_{ab}^{(2)} = \frac{2}{4\tilde{f}_0^2 \tilde{f}^2} \frac{4}{N_f} \sum_c \text{tr}(t^a T^c) \text{tr}(t^b T^c) = \frac{2\delta_{ab}}{N_f \tilde{f}_0^2 \tilde{f}^2}, \quad (6.11)$$

where in the first equation the factor of 2 reflects the two possibilities above and in the second equation we have used  $\text{tr}(t^a T^c) = \delta_{ac}$ .

3. For  $A \neq 0$  and  $B \neq 0$ , the contribution to  $Q$  can be obtained from [84, eq. (48)] by replacing  $2N_f \rightarrow N_f$  and multiplying by 8 to correct for the difference in the normalization of the generators (which in [84] is  $\text{tr}(t^a t^b) = \delta_{ab}/2$ ). This yields

$$Q_{ab}^{(3)} = \delta_{ab} \frac{(N_f - 4)(N_f + 2)}{2N_f \tilde{f}^4}. \quad (6.12)$$

Summing up these contributions,  $Q_{ab}$  is given by

$$Q_{ab} = Q_{ab}^{(1)} + Q_{ab}^{(2)} + Q_{ab}^{(3)} = \delta_{ab} \left[ \frac{(N_f - 4)(N_f + 2)}{2N_f \tilde{f}^4} + \frac{2}{N_f \tilde{f}_0^2 \tilde{f}^2} \right], \quad (6.13)$$

and our final result for the scalar susceptibility from the effective theory is

$$K_{ab}(j) \sim \delta_{ab} \left[ \frac{(N_f - 4)(N_f + 2)}{2N_f \tilde{f}^4} + \frac{2}{N_f \tilde{f}_0^2 \tilde{f}^2} \right] \frac{\Phi_{\mathbf{H}}^2}{16\pi^2} \ln \left( \frac{\tilde{\Lambda}}{j} \right). \quad (6.14)$$

---

<sup>38</sup>This is a valid approximation as we are interested in the infrared limit of the theory.

Now let us compare (6.4) and (6.14). First of all, note that the constant part  $\rho_{\text{sv}}(0)$  in (6.4) does not contribute to  $K_{ab}$  since

$$\int_0^\infty d\xi \frac{\xi^2 - j^2}{(\xi^2 + j^2)^2} = 0. \quad (6.15)$$

Hence only the difference  $\rho_{\text{sv}}(\xi) - \rho_{\text{sv}}(0)$  is relevant. To reproduce the singularity  $\sim \ln(\tilde{\Lambda}/j)$  in (6.14), we must have

$$\rho_{\text{sv}}(\xi) - \rho_{\text{sv}}(0) = C\xi \quad \text{for } \xi > 0 \quad (6.16)$$

in the vicinity of  $\xi = 0$ , where  $C = \rho'_{\text{sv}}(0)$ . Now,

$$\int_0^{\tilde{\Lambda}} d\xi \frac{C\xi(\xi^2 - j^2)}{(\xi^2 + j^2)^2} \sim C \ln\left(\frac{\tilde{\Lambda}}{j}\right), \quad (6.17)$$

and thus the slope of the singular value density at the origin is given by

$$\rho'_{\text{sv}}(0) = \left[ \frac{(N_f - 4)(N_f + 2)}{2N_f \tilde{f}^4} + \frac{2}{N_f \tilde{f}_0^2 \tilde{f}^2} \right] \frac{\Phi_{\mathbf{H}}^2}{16\pi^2}. \quad (6.18)$$

Note that  $\Phi_{\mathbf{H}}$ ,  $\tilde{f}_0$ , and  $\tilde{f}$  are functions of  $\mu$ . To what extent this result is still valid at  $\mu < \infty$  will be discussed in section 6.4.

For  $N_f = 2$  the Smilga-Stern method used above does not work since in (6.1) we then have  $J = jI$  so that  $K_{ab}(j)$  cannot be defined as in (6.2). A similar problem occurs in the derivation of the slope of the Dirac eigenvalue density, where the Smilga-Stern method fails for  $N_f = 1$ . In that case the slope could still be computed using partially quenched perturbation theory [84], and it was found that the result obtained from the Smilga-Stern method remains valid for  $N_f = 1$ . It is therefore tempting to speculate that (6.18) remains valid for  $N_f = 2$ , but to confirm this we would have to compute  $\rho_{\text{sv}}(\xi)$  in partially quenched perturbation theory. Such a rather complicated calculation is deferred to future work.

Let us add two comments here. First, we could relax the assumption  $J_R = -J_L$ , and in particular we could set  $J_R$  (or  $J_L$ ) to zero.<sup>39</sup> This would give us the slope of the density of the left-handed (or right-handed) singular values, which is 1/2 of the full slope because the factor of 2 mentioned after (6.9) would be absent. Second, we have now computed  $\rho_{\text{sv}}(0)$  and  $\rho'_{\text{sv}}(0)$  and therefore obtained information on the singular value density near zero. An analytical result can also be computed for asymptotically large  $\xi$ , which, owing to asymptotic freedom, can be described by the free theory without coupling to the gauge field. Thus the whole singular value spectrum can be understood at least qualitatively by an interpolation of two tractable limits. We obtain (for an arbitrary number  $N_c \geq 2$  of colors)

$$\rho_{\text{sv}}(\xi) \rightarrow \frac{N_c}{2\pi^2} \xi(\xi^2 + 2\mu^2) \quad \text{for } \xi \rightarrow \infty. \quad (6.19)$$

For  $\mu \rightarrow 0$  this reduces to twice the Dirac eigenvalue density in the free limit, as expected. An outline of the derivation is given in appendix E.

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<sup>39</sup>This results in a projection on the topologically trivial sector (see (2.10)), which is immaterial in the  $p$ -regime.

## 6.2 Intermediate density

The calculation at intermediate density is very similar to that at infinite density. Since the fundamental microscopic theory is unchanged, equations (6.1) through (6.4) also remain unchanged. On the effective theory side, (6.5) is replaced by

$$Z_{\text{eff}}^{\mathbf{I}} = \int \mathcal{D}\Sigma_L \mathcal{D}\Sigma_R \mathcal{D}V \exp \left( - \int d^4x \mathcal{L}_{\text{eff}}^{\mathbf{I}} \right). \quad (6.20)$$

We could now go through a similar calculation as in section 6.1, using (5.24) through (5.26). However, it is easier to note that the only difference is the replacement of the two U(1) fields  $L$  and  $R$  by a single U(1) field  $V$ . The only contributions of the U(1) fields to  $Q_{ab}$  are in  $Q_{ab}^{(2)}$ , and it follows from the calculation in section 6.1 that we can obtain  $Q_{ab}^{(2)}$  for the present case by dividing the result in (6.11) by 2. Everything else remains unchanged so that the slope for  $N_f \geq 4$  is now given by

$$\rho'_{\text{sv}}(0) = \left[ \frac{(N_f - 4)(N_f + 2)}{2N_f f^4} + \frac{1}{N_f f_0^2 f^2} \right] \frac{\Phi_{\mathbf{I}}^2}{16\pi^2}. \quad (6.21)$$

Again,  $\Phi_{\mathbf{I}}$ ,  $f_0$ , and  $f$  are functions of  $\mu$ . It is tempting to speculate that (6.21) remains valid for  $N_f = 2$ .

## 6.3 Zero density

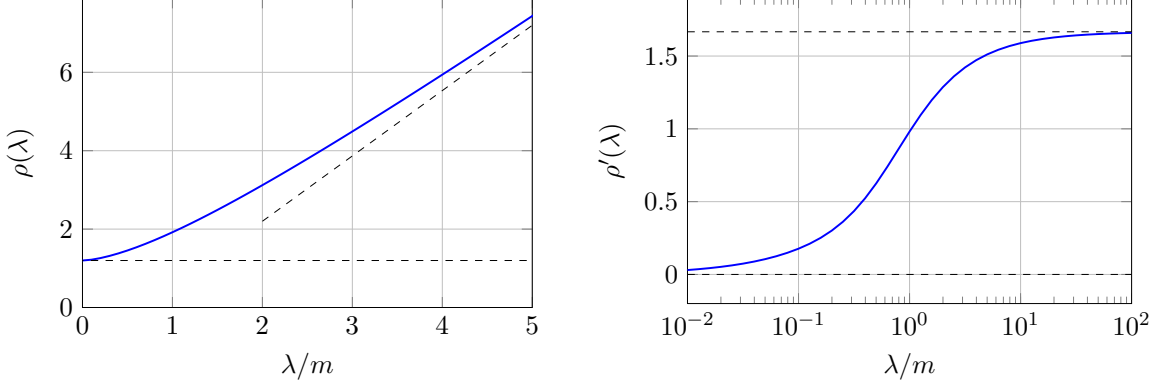
Let us now consider strictly zero chemical potential. Again, equations (6.1) through (6.4) remain unchanged, but the coset space of the effective theory is now  $\text{SU}(2N_f)/\text{Sp}(2N_f)$ . It follows from the calculation in section 6.1 that  $Q_{ab}$  is now entirely given by  $Q_{ab}^{(3)}$  and that we can obtain  $Q_{ab}^{(3)}$  for the present case from the result in (6.12) by replacing  $N_f \rightarrow 2N_f$ ,  $\tilde{f} \rightarrow F$ , and dividing by 2 since the left- and right-handed modes are already contained in  $\text{SU}(2N_f)/\text{Sp}(2N_f)$ . This yields for  $N_f \geq 2$

$$\rho'_{\text{sv}}(0) = \frac{(N_f - 2)(N_f + 1)}{N_f F^4} \frac{\Phi_{\mathbf{L}}^2}{16\pi^2}, \quad (6.22)$$

where  $\Phi_{\mathbf{L}}$  and  $F$  are now independent of  $\mu$ . To what extent this result is still valid at nonzero  $\mu$  will be discussed in the next subsection.

## 6.4 Relation between the three results

In the previous three subsections we have obtained three different results for the slope  $\rho'_{\text{sv}}(0)$  for  $\mu = \infty$ , intermediate  $\mu$ , and  $\mu = 0$ , respectively. At first glance it does not seem possible to interpolate them smoothly, and we thus encounter a puzzle: How are the three results related? What actually happens to the spectrum if  $\mu$  is continuously changed? Below we argue that there is no puzzle here. To simplify the presentation we divide our discussion into three parts, the first one for three-color QCD at  $\mu = 0$  as an instructive model case for our problem, the second one for the low-density region, and the third one for the high-density region of two-color QCD. The arguments are analogous, however.



**Figure 5.** Left: Dirac eigenvalue density in three-color QCD at  $\mu = 0$  for two massless flavors and one flavor with mass  $m$ , in units of  $m\Sigma^2/32\pi^2F^4$ . The intercept with the vertical axis is arbitrary due to renormalization [83]. Right: Slope of the density in units of  $\Sigma^2/32\pi^2F^4$ . The dotted lines in both plots correspond to the slopes 0 ( $N_f = 2$ ) and  $5/3$  ( $N_f = 3$ ).

#### 6.4.1 Zero density ( $\beta = 2$ ): A journey from $N_f = 2$ to 3

So far we have argued that there are three effective theories for dense two-color QCD and that as a function of  $\mu$  they change smoothly from one to another. This situation has an exact counterpart in three-color QCD for  $N_f = 3$  at  $\mu = 0$ , where the strange quark mass serves as a “knob” to interpolate between the chiral perturbation theories for  $N_f = 2$  and  $N_f = 3$ . Therefore we study this simpler case first before considering the more exotic case of dense two-color QCD.

The original Smilga-Stern relation [2], derived for the Dirac eigenvalue density (not the singular value density) in three-color QCD ( $\beta = 2$ ) at  $\mu = 0$  with  $N_f$  flavors in the chiral limit, reads

$$\rho(\lambda) = \frac{\Sigma}{\pi} + \frac{\Sigma^2}{32\pi^2F^4} \frac{N_f^2 - 4}{N_f} |\lambda| + o(\lambda) \quad (6.23)$$

with the chiral condensate  $\Sigma$  and the pion decay constant  $F$ . The coefficient of  $|\lambda|$  depends on  $N_f$ . For example, the slope vanishes for  $N_f = 2$  but is nonzero for  $N_f = 3$ . It is not clear from this expression alone how the slope changes if we add a nonzero strange quark mass to change the number of light flavors continuously.

The generalization of (6.23) to nonzero degenerate masses was given in [82]<sup>40</sup> and later extended to nondegenerate sea quark masses in [83]. Therefore we can employ the results of [83] to study the density  $\rho(\lambda)$  and the slope  $\rho'(\lambda)$  for  $N_f = 3$  with a massive strange quark. Setting  $(m_1, m_2, m_3) = (0, 0, m)$  in [83, Eq. (17)] we plot  $\rho(\lambda)$  and  $\rho'(\lambda)$  in figure 5 as a function of  $\lambda/m$ .

The curves nicely interpolate between two limits: For  $\lambda \ll m$  the strange quark is heavy relative to the probed scale and we get the slope for  $N_f = 2$ :  $\rho'(\lambda) \propto (N_f^2 - 4)/N_f = 0$ . In the limit  $m \ll \lambda$  ( $\ll \Lambda \sim 4\pi F$ ) the strange quark is light relative to the probed scale and

<sup>40</sup>There is a typo in [82, Eq. (84)]. The second term in the square brackets of that equation must be multiplied by  $\pi$ , as is evident from [82, Eq. (83)].

we get the slope for  $N_f = 3$ :  $\rho'(\lambda) \propto (N_f^2 - 4)/N_f = 5/3$ . The transition occurs smoothly around  $\lambda \sim m$ . Thus we can draw the conclusion that no contradiction arises from different values of the slopes for  $N_f = 2$  and  $N_f = 3$ , because at nonzero  $m$  they correspond to different domains of the spectrum.

This finding can be interpreted within partially quenched chiral perturbation theory as follows. In this method we add valence flavors and compute the spectral density from the valence quark mass dependence of the chiral condensate [85]. We therefore deal with two classes of mesons, one being made of only sea quarks, and the other being made of valence quarks (and sea quarks). If the latter (“valence mesons”) are much heavier than the former (“sea mesons”), i.e.,  $\lambda \gg m$ , then all three sea flavors contribute to the valence quark mass dependence of the chiral condensate, implying  $N_f = 3$ . Conversely, if  $\lambda \ll m$ , the heavier sea mesons are decoupled, reducing the computation to  $N_f = 2$ . The transition between these two cases occurs around  $\lambda \sim m$ , i.e., when the masses of the sea and valence mesons are roughly equal.

#### 6.4.2 Low density

In this subsection we discuss the relation between the results (6.21) and (6.22) for  $\mu \ll \Lambda$ . The chemical potential plays exactly the same role as the strange quark mass in the previous subsection, both acting as explicit symmetry-breaking parameters. This analogy is the basis of our following argument.

We first note that  $\rho_{\text{sv}}(\xi)$  can be computed in partially quenched chiral perturbation theory, starting from

$$Z_{N_f+2|2}(j; j_v, j'_v) = \left\langle \det^{N_f/2}(D^\dagger D + j^2) \frac{\det(D^\dagger D + j_v^2)}{\det(D^\dagger D + j_v'^2)} \right\rangle_{\text{YM}} \quad (6.24)$$

and setting  $j_v = j'_v = i\xi + \varepsilon$  (with  $\varepsilon \rightarrow 0^+$ ) at the end of the calculation. We will not actually perform this computation but use (6.24) for qualitative estimates. In comparison to the usual setting [82, 84],  $j$  and  $j_v, j'_v \sim \xi$  correspond to the sea and valence quark masses, respectively. In the following  $j$  is always assumed to be infinitesimal. Our main concern is the competition between  $\mu$  and  $j_v$ .

We can now replace  $Z_{N_f+2|2}(j; j_v, j'_v)$  by an effective partition function formulated in terms of NG modes. We have two options, either the partially quenched extension of theory **L**, or that of theory **I**. Let us discuss them separately.

- **Theory I:** For the partially quenched extension of theory **I** to be valid, the condition (i) of section 5.5 must be satisfied for all NG modes, i.e., their masses must be much smaller than  $m_\ell$ . The new ingredient here is that, in addition to the NG modes discussed in section 5.4, we now also have NG modes containing the valence quarks corresponding to  $j_v$  and  $j'_v$ , which we call valence NG modes. At low density their masses are of order  $\sqrt{j_v \Lambda} \sim \sqrt{\xi \Lambda}$ , see (5.10a) with  $F \sim \Phi_{\mathbf{L}}^{1/3} \sim \Lambda$ , and hence the condition is

$$\xi \ll \frac{\mu^2}{\Lambda}. \quad (6.25)$$

In this domain the slope  $\rho'_{\text{sv}}(\xi)$  is given by (6.21) at leading order of the low-energy expansion.

- **Theory L:** For the partially quenched extension of theory **L** to be valid, the masses of all NG modes must again be much smaller than  $m_\ell$ . This time the masses of the valence NG modes are of order  $\sqrt{j_v\Lambda} \sim \sqrt{\xi\Lambda}$  and  $\sqrt{j_v\Lambda + \mu^2} \sim \sqrt{\xi\Lambda + \mu^2}$ , see (5.10) with  $F \sim \Phi_{\mathbf{L}}^{1/3} \sim \Lambda$ . The condition is therefore

$$\xi \ll \Lambda. \quad (6.26)$$

This is not end of the story, however: Theory **L** is more complicated than theory **I**, because we have two scales  $\mu$  and  $\sqrt{j_v\Lambda}$  (analogous to  $m$  and  $\lambda$  in section 6.4.1) whose ratio controls the final result in a nontrivial way. Based on our experience in section 6.4.1 we expect the following.

- For  $\mu \ll \sqrt{j_v\Lambda} \sim \sqrt{\xi\Lambda}$ : All sea NG modes contribute, and the result for  $\rho'_{\text{sv}}(\xi)$  agrees with the result (6.22) at  $\mu = 0$
- For  $\mu \gg \sqrt{j_v\Lambda} \sim \sqrt{\xi\Lambda}$ : The NG modes with masses of order  $\mu$  decouple from the computation of  $\rho_{\text{sv}}(\xi)$ , and theory **L** reduces to theory **I** in which the heavy modes have been integrated out. The slope thus agrees with (6.21) from theory **I**.

Putting everything together, we see that in the regime  $\mu \ll \Lambda$  the results from theory **I** and **L** for the slope  $\rho'_{\text{sv}}(\xi)$  are valid in the following domains:

$$(6.21) \text{ from } \mathbf{I}: \quad \xi \ll \frac{\mu^2}{\Lambda}, \quad (6.27)$$

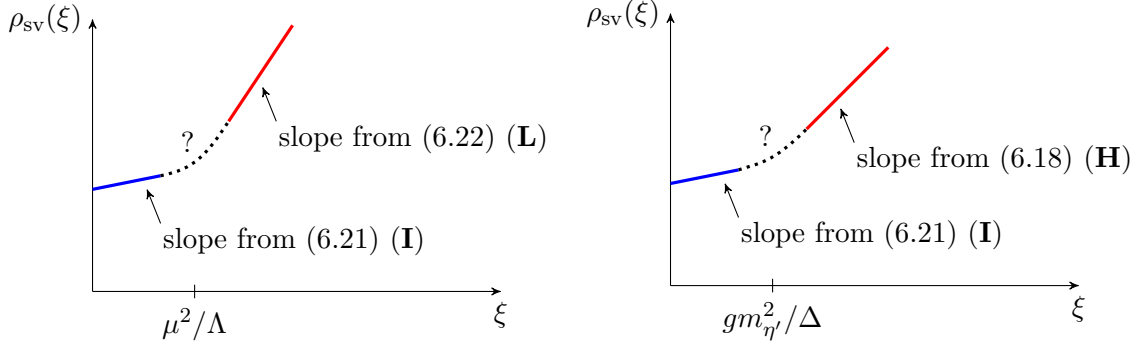
$$(6.22) \text{ from } \mathbf{L}: \quad \frac{\mu^2}{\Lambda} \ll \xi \ll \Lambda. \quad (6.28)$$

These findings are illustrated in figure 6 (left). The slope is first given by (6.21), and we conjecture that it changes smoothly to the value given by (6.22). To avoid confusion we point out that for  $\mu = 0$  the window in which the slope is given by (6.21) shrinks to zero so that the slope at the origin is given by (6.22).

We now present a nontrivial cross-check of our conclusion that does not hinge on partially quenched chiral perturbation theory. Let us return to our discussions in sections 6.1–6.3 without valence quarks and consider taking the limits  $\mu \rightarrow 0$  and  $j \rightarrow 0$  while keeping the condition  $\mu^2 \gg j\Lambda$ . In this case the two types of NG modes have masses of order  $\sqrt{j\Lambda}$  (lighter) and  $\mu$  (heavier), respectively. The one-loop integral (6.7) in the effective theory can be evaluated for the cases of two lighter, two heavier, or one lighter and one heavier NG modes circulating around the loop. For these cases we obtain

$$\int^\Lambda d^4p \frac{1}{(p^2 + j\Lambda)^2} \sim \ln \frac{\Lambda}{j}, \quad (6.29)$$

$$\int^\Lambda d^4p \frac{1}{(p^2 + \mu^2)^2} \sim \ln \frac{\Lambda^2}{\mu^2}, \quad (6.30)$$



**Figure 6.** The behavior of the singular value density near zero. Left: low density ( $\mu \ll \Delta$ ), right: high density (so that  $m_{\eta'} \ll \Delta$ ), see text for details.

$$\int^\Lambda d^4p \frac{1}{(p^2 + j\Lambda)(p^2 + \mu^2)} \sim \frac{\mu^2 \ln \frac{\Lambda^2}{\mu^2} - j\Lambda \ln \frac{\Lambda}{j}}{\mu^2 - j\Lambda} \sim \ln \frac{\Lambda^2}{\mu^2} \quad \text{for } \mu^2 \gg j\Lambda. \quad (6.31)$$

Therefore the sum of all one-loop contributions to  $K_{ab}$  is given by

$$\alpha \ln \frac{\Lambda}{j} + \beta \ln \frac{\Lambda^2}{\mu^2} \quad (6.32)$$

with prefactors  $\alpha$  and  $\beta$  that also contain traces of the generators. The two infrared singularities in (6.32) (generated by  $j \rightarrow 0$  and  $\mu \rightarrow 0$ , respectively) must be matched by corresponding singularities in the microscopic theory. Motivated by figure 6, let us assume that  $\rho_{\text{sv}}(\xi)$  can be approximated by a straight line with slope  $\alpha$  for  $\xi < \xi_c$  and by another straight line with slope  $\alpha + \beta$  for  $\xi > \xi_c$ , with  $\xi_c$  an unknown function of  $\mu$ . Then (6.4) becomes

$$\begin{aligned} & \int_0^\Lambda d\xi \rho_{\text{sv}}(\xi) \frac{\xi^2 - j^2}{(\xi^2 + j^2)^2} \\ &= \int_0^{\xi_c} d\xi (\alpha\xi + \text{const.}) \frac{\xi^2 - j^2}{(\xi^2 + j^2)^2} + \int_{\xi_c}^\Lambda d\xi [(\alpha + \beta)\xi + \text{const.}] \frac{\xi^2 - j^2}{(\xi^2 + j^2)^2} \\ &\sim \alpha \ln \frac{\xi_c}{j} + (\alpha + \beta) \ln \frac{\Lambda}{\xi_c} = \alpha \ln \frac{\Lambda}{j} + \beta \ln \frac{\Lambda}{\xi_c}. \end{aligned} \quad (6.33)$$

Matching (6.32) and (6.33) yields

$$\xi_c \sim \frac{\mu^2}{\Lambda}, \quad (6.34)$$

in agreement with the argument based on partially quenched chiral perturbation theory.

### 6.4.3 High density

We now clarify the relation between (6.18) from theory **H** and (6.21) from theory **I** at large  $\mu$ . The arguments are analogous to those at small  $\mu$ , except that  $m_{\text{inst}}$  now plays the role of  $\mu$  as an external symmetry-breaking parameter. First of all we require that the masses of all NG modes (sea and valence) must be sufficiently below  $m_\ell$ . At high density the masses

of the valence NG modes in the partially quenched theory are of order  $\sqrt{j_v \Delta/g} \sim \sqrt{\xi \Delta/g}$  and  $\sqrt{j_v \Delta/g + m_{\text{inst}}^2} \sim \sqrt{\xi \Delta/g + m_{\text{inst}}^2}$ , see (5.20) with  $\tilde{f}_A \sim \mu$  [74] and  $\Phi_{\mathbf{H}} \sim \mu^2 \Delta/g$  [81], where  $g$  is the running coupling constant. Using the relevant results for  $m_\ell$  in table 1 we obtain the following bounds on the values of  $\xi$  below which  $\rho_{\text{sv}}(\xi)$  can be computed from the partially quenched extensions of the effective theories **I** or **H** in the regime where  $m_{\eta'} \ll \Delta$ :

$$\xi \ll \frac{gm_{\eta'}^2}{\Delta} \quad \text{for theory } \mathbf{I}, \quad (6.35)$$

$$\xi \ll g\Delta \quad \text{for theory } \mathbf{H}. \quad (6.36)$$

Thus  $\rho'_{\text{sv}}(\xi)$  is given by (6.21) from theory **I** in the range (6.35). We note that the scale  $gm_{\eta'}^2/\Delta$  goes to zero rapidly as  $\mu \rightarrow \infty$ .

On the other hand, theory **H** has two scales,  $\sqrt{j_v \Delta/g}$  and  $m_{\text{inst}}$ . A rerun of the arguments at small  $\mu$  then shows that in the regime where  $m_{\eta'} \ll \Delta$  the slope  $\rho'_{\text{sv}}(\xi)$  is given by the results (6.18) and (6.21) in the following domains,

$$(6.21) \text{ from } \mathbf{I} : \quad \xi \ll \frac{gm_{\eta'}^2}{\Delta}, \quad (6.37)$$

$$(6.18) \text{ from } \mathbf{H} : \quad \frac{gm_{\eta'}^2}{\Delta} \ll \xi \ll g\Delta. \quad (6.38)$$

This is illustrated in figure 6 (right). For  $\mu \rightarrow \infty$  the window in which the slope is given by (6.21) shrinks to zero so that the slope at the origin is given by (6.18).

The second argument presented in section 6.4.2 works in exactly the same way here and leads to

$$\xi_c \sim \frac{gm_{\eta'}^2}{\Delta} \quad (6.39)$$

as expected.

Note that the result (6.21) is actually valid for all  $0 < \mu < \infty$ . We can replace  $\rho'_{\text{sv}}(0)$  by  $\rho'_{\text{sv}}(\xi)$  in (6.21) for sufficiently small  $\xi$ , the upper bound of which is given in (6.34) and (6.39) at small and large  $\mu$ , respectively. At intermediate density we do not have an estimate for the upper bound because  $m_\ell$  is unknown in this region.

So far we have explained what we believe is the most reasonable behavior of the singular value density at nonzero  $\mu$  based on the partial quenching technique and the analogy to three-color QCD at  $\mu = 0$  with a heavy strange quark. For a solid proof of our conjecture shown in figure 6 one would have to compute the slope in partially quenched chiral perturbation theory explicitly, but this is beyond the scope of this paper. It would be interesting to check our new Smilga-Stern-type relations by lattice simulations. This is possible in principle as the infamous sign problem is absent in this theory.



## 7 Finite-volume analysis: Leutwyler-Smilga-type sum rules and random matrix theories ( $\beta = 1$ )

### 7.1 The $\varepsilon$ -regime

In this section we study two-color QCD with diquark sources in a finite volume  $V_4 = L^4$  (and again in the chiral limit). As in QCD there is a regime, the so-called  $\varepsilon$ -regime [3], in which the kinetic terms in the effective chiral Lagrangian can be neglected so that the theory becomes zero-dimensional and the partition function is dominated by the zero-momentum modes of the NG particles. The condition for the  $\varepsilon$ -regime is

$$\frac{1}{m_\ell} \ll L \ll \frac{1}{m_{\text{NG}}}, \quad (7.1)$$

where  $m_\ell$  is again the mass scale of the lightest non-NG particle and  $m_{\text{NG}}$  is the mass scale of the NG particles that are included in the effective theory. The first inequality in (7.1) means that the contribution of the non-NG particles to the partition function can be neglected, while the second inequality means that the Compton wavelength of the NG particles is larger than the size of the box, which in turn implies that the functional integral over the NG fields can be replaced by a zero-mode integral over the coset space parametrized by them. For the three different effective theories in section 5 the values of  $m_\ell$  are given in section 5.5, and the  $m_{\text{NG}}$  are given in (5.10), (5.20), and (5.27). Note that for zero diquark sources and finite  $L$  the second inequality in (7.1) is always satisfied in theory **I** since  $m_{\text{NG}} = 0$ . Note also that while the domains of validity of the effective theories overlap (see section 5.5), this is not the case for the corresponding  $\varepsilon$ -regimes: At low density it follows from (7.1) that  $1/\mu \ll L$  for **I** and  $L \ll 1/\mu$  for **L**, and these conditions are mutually exclusive. Similarly, at high density we have  $1/m_{\eta'} \ll L$  for **I** and  $L \ll 1/m_{\eta'}$  for **H**.

At  $\mu = 0$ , it is well known that there exists a scale  $E_T$  below which the eigenvalue spectrum of the Dirac operator obeys chiral random matrix theory [82, 86]. The scale  $E_T$  is called Thouless energy, borrowing the nomenclature from mesoscopic physics. The equivalence between random matrix theory and the zero-momentum limit of the partially quenched effective theory shows that  $E_T$  can be understood as the energy above which the condition (7.1) no longer holds in the partially quenched theory and the modes with nonzero momentum start to contribute.

We now comment on the Thouless energy for the singular value spectrum at  $0 < \mu < \infty$ , based on the partially quenched extension of the three effective theories introduced in section 6.4.2. For simplicity we let  $j = 0$  in (6.24) and concentrate on the “spectral mass”  $j_v$ . We assume that the condition  $L \gg 1/m_\ell$  is satisfied for all cases considered below.

- Theory **I** ( $0 < \mu < \infty$ ): From (5.27) we find that the masses of the valence NG modes are given by  $m_{\text{vNG}}^2 \sim j_v \Phi_{\mathbf{I}}/f^2 \sim \xi \Phi_{\mathbf{I}}/f^2$ , where all low-energy constants depend on  $\mu$  implicitly.<sup>41</sup> For  $m_{\text{vNG}} \ll 1/L$  the  $j_v$ -dependence of the partition function is governed

<sup>41</sup>We are sloppy about the distinction between  $f$  and  $f_0$  here, but they are of the same order of magnitude so that the distinction does not change our discussion.

by the NG modes with zero momentum. Thus the Thouless energy is determined by

$$\sqrt{E_T \Phi_{\mathbf{I}}/f^2} = \frac{1}{L} \quad \rightarrow \quad E_T = \frac{f^2}{L^2 \Phi_{\mathbf{I}}}. \quad (7.2)$$

- Theory **H** ( $m_{\eta'} \ll \Delta$ ): From (5.20) we have  $m_{\text{vNG}}^2 \sim j_v \Phi_{\mathbf{H}}/\tilde{f}^2 \sim j_v \Delta/g$  (see section 6.4.3) and  $j_v \Phi_{\mathbf{H}}/\tilde{f}^2 + m_{\text{inst}}^2 \sim j_v \Delta/g + m_{\text{inst}}^2$ . Assuming  $m_{\text{inst}} \ll 1/L$  we obtain

$$\sqrt{E_T \Delta/g} = \frac{1}{L} \quad \rightarrow \quad E_T = \frac{g}{L^2 \Delta}. \quad (7.3)$$

- Theory **L** ( $\mu \ll \Lambda$ ): From (5.10) we have  $m_{\text{vNG}}^2 \sim j_v \Phi_{\mathbf{L}}/F^2$  and  $j_v \Phi_{\mathbf{L}}/F^2 + (2\mu)^2$ , where this time the low-energy constants are  $\mu$ -independent. Assuming  $\mu \ll 1/L$  we obtain

$$\sqrt{E_T \Phi_{\mathbf{L}}/F^2} = \frac{1}{L} \quad \rightarrow \quad E_T = \frac{F^2}{L^2 \Phi_{\mathbf{L}}}. \quad (7.4)$$

Thus in all cases  $E_T \propto 1/\sqrt{V_4}$ . Essentially, this is due to the fact that the diquark source enters the effective Lagrangian linearly at any  $\mu$ . This completes our discussion of the Thouless energy.

In the  $\varepsilon$ -regime we can compute exact sum rules for the inverse singular values of the Dirac operator. This will be done in the following three subsections for the three different density regimes. For technical reasons we start with intermediate density this time. We will also derive chiral random matrix theories that allow us to compute microscopic correlation functions of the singular values.<sup>42</sup> The sum rules are simply moments of these correlation functions. The results for  $\rho_{\text{sv}}(\xi)$  from the random matrix theories are valid in the range  $\xi \ll E_T$ .

## 7.2 Intermediate density

In analogy to the analysis of Leutwyler and Smilga [4] we first project the partition function onto sectors of fixed topological charge and then expand it in powers of the diquark sources. This is done both in the microscopic theory and in the effective theory. Matching the coefficients of the sources then yields sum rules for the inverse singular values.

On the QCD side, we start from (2.10) and assume that there are no accidental zero modes. Without loss of generality we therefore set  $n_L = 0$  and  $n_R = \nu \geq 0$ .<sup>43</sup> We also introduce the notation

$$\langle O \rangle_\nu = \frac{\langle O \det'(D^\dagger D)^{N_f/2} \rangle_{\text{YM}, \nu}}{\langle \det'(D^\dagger D)^{N_f/2} \rangle_{\text{YM}, \nu}}, \quad (7.5)$$

where the subscript  $\nu$  indicates that the average is only over gauge fields with fixed topological charge  $\nu$  and the prime, as always, means that the zero modes are omitted. Using

$$\det'(D^\dagger D + J_L^\dagger J_L P_L + J_R^\dagger J_R P_R) = \prod_n' \det(\xi_{Ln}^2 + J_L^\dagger J_L)(\xi_{Rn}^2 + J_R^\dagger J_R) \quad (7.6)$$

<sup>42</sup>In addition, a chiral random matrix theory for QCD with isospin chemical potential ( $\beta = 2$ ) will be derived in appendix F.

<sup>43</sup>As usual, for  $\nu < 0$  the final results of this section remain valid, except that  $\nu$  must be replaced by  $|\nu|$ .

the partition function for fixed topology is given by

$$\frac{Z_\nu(J_L, J_R)}{\langle \det'(D^\dagger D)^{N_f/2} \rangle_{\text{YM}, \nu}} = [\text{Pf}(J_L^\dagger) \text{Pf}(J_R)]^\nu \left\langle \prod'_n \det^{1/2} \left( \mathbb{1} + \frac{J_L^\dagger J_L}{\xi_{Ln}^2} \right) \left( \mathbb{1} + \frac{J_R^\dagger J_R}{\xi_{Rn}^2} \right) \right\rangle_\nu. \quad (7.7)$$

Using the formula

$$\det(\mathbb{1} + \varepsilon) = 1 + \text{tr} \varepsilon + \frac{1}{2}[(\text{tr} \varepsilon)^2 - \text{tr}(\varepsilon^2)] + O(\varepsilon^3) \quad (7.8)$$

we expand  $Z_\nu$  in powers of the diquark sources,

$$\begin{aligned} & \frac{Z_\nu(J_L, J_R)}{\langle \det'(D^\dagger D)^{N_f/2} \rangle_{\text{YM}, \nu} [\text{Pf}(J_L^\dagger) \text{Pf}(J_R)]^\nu} \\ &= 1 + \left[ \frac{\text{tr} J_L^\dagger J_L}{2} \left\langle \sum'_n \frac{1}{\xi_{Ln}^2} \right\rangle_\nu - \frac{\text{tr}(J_L^\dagger J_L)^2}{4} \left\langle \sum'_n \frac{1}{\xi_{Ln}^4} \right\rangle_\nu + \frac{(\text{tr} J_L^\dagger J_L)^2}{8} \left\langle \left( \sum'_n \frac{1}{\xi_{Ln}^2} \right)^2 \right\rangle_\nu \right. \\ & \quad \left. + (L \leftrightarrow R) \right] + \frac{\text{tr} J_L^\dagger J_L \text{tr} J_R^\dagger J_R}{4} \left\langle \left( \sum'_n \frac{1}{\xi_{Ln}^2} \right) \left( \sum'_m \frac{1}{\xi_{Rm}^2} \right) \right\rangle_\nu + O(J^6). \end{aligned} \quad (7.9)$$

On the low-energy effective theory side, let us first consider the case  $N_f \geq 4$ . We start from (5.24) and neglect the kinetic terms to obtain the finite-volume partition function

$$Z^{\text{eff}}(J_L, J_R) = \int d\Sigma_L d\Sigma_R dV \exp [V_4 \Phi_{\mathbf{I}} \text{Re tr}(J_L \Sigma_L - J_R \Sigma_R) V], \quad (7.10)$$

where the integrals over  $\Sigma_{L,R}$  and  $V$  are no longer functional integrals but simple integrals over  $\text{SU}(N_f)/\text{Sp}(N_f)$  and  $\text{U}(1)$ , respectively. It is convenient to define

$$\tilde{J}_i = J_i V_4 \Phi_{\mathbf{I}} \quad (i = L, R). \quad (7.11)$$

As explained after (2.12), a nonzero  $\theta$ -angle can be introduced by redefining, e.g.,  $J_L \rightarrow J_L e^{-i\theta/N_f}$  and  $J_R \rightarrow J_R e^{i\theta/N_f}$ . To project onto topological sectors we note that the inversion of (2.12) is

$$Z_\nu = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\nu\theta} Z(\theta), \quad (7.12)$$

which in the present case gives

$$\begin{aligned} Z_\nu^{\text{eff}}(J_L, J_R) &= \int d\Sigma_L d\Sigma_R dV \frac{d\theta}{2\pi} \exp [-i\nu\theta + \text{Re tr}(\tilde{J}_L e^{-i\theta/N_f} \Sigma_L - \tilde{J}_R e^{i\theta/N_f} \Sigma_R) V] \\ &= \int d\Sigma_L d\Sigma_R dV \frac{d\theta}{2\pi} \exp [-iN_f \nu \theta + \text{Re tr}(\tilde{J}_L e^{-i\theta} \Sigma_L - \tilde{J}_R e^{i\theta} \Sigma_R) V] \\ &= \int d\Sigma_L d\Sigma_R dL dR (LR^\dagger)^{\frac{1}{2}N_f \nu} \exp [\text{Re tr}(\tilde{J}_L L \Sigma_L - \tilde{J}_R R \Sigma_R)], \end{aligned} \quad (7.13)$$

where in the second step we have redefined  $\theta \rightarrow N_f \theta$  and used the periodicity of the  $\theta$ -integral to extend the integration region from  $[0, 2\pi/N_f]$  back to  $[0, 2\pi]$ . In the last step we have introduced the new  $\text{U}(1)$  integration variables

$$L = e^{-i\theta} V \quad \text{and} \quad R = e^{i\theta} V. \quad (7.14)$$

Parametrizing  $\Sigma_i = U_i I U_i^T$  with  $U_i \in \text{SU}(N_f)/\text{Sp}(N_f)$  as in (5.11) we note that

$$\text{Re tr}(\tilde{J}_R R \Sigma_R) = \text{Re tr}(\tilde{J}_R^\dagger R^\dagger \Sigma_R^\dagger) = \text{Re tr}(-\tilde{J}_R^\dagger R^\dagger U_R^* I U_R^\dagger). \quad (7.15)$$

Redefining the integration variables  $R \rightarrow R^\dagger$  and  $U_R \rightarrow U_R^*$  and combining  $L$  and  $R$  with  $\Sigma_L$  and  $\Sigma_R$  as in (5.18) we thus obtain from (7.13)

$$\begin{aligned} Z_\nu^{\text{eff}}(J_L, J_R) &= \int d\Sigma_L d\Sigma_R dL dR (LR)^{\frac{1}{2}N_f\nu} \exp[\text{Re tr}(\tilde{J}_L L \Sigma_L + \tilde{J}_R^\dagger R \Sigma_R)] \\ &= \int d\tilde{\Sigma}_L d\tilde{\Sigma}_R \det^{\nu/2}(\tilde{\Sigma}_L \tilde{\Sigma}_R) \exp[\text{Re tr}(\tilde{J}_L \tilde{\Sigma}_L + \tilde{J}_R^\dagger \tilde{\Sigma}_R)]. \end{aligned} \quad (7.16)$$

Recalling  $\tilde{\Sigma}_i = U_i I U_i^T$  with  $U_i \in \text{U}(N_f)/\text{Sp}(N_f)$  we note that we can extend the integration to be over  $U_i \in \text{U}(N_f)$  because the generators of  $\text{Sp}(N_f)$  leave  $I$  invariant and therefore drop out of the combination  $U_i I U_i^T$ . We now define, for a complex antisymmetric matrix  $X$  of even dimension  $N_f$ , the function [87]

$$g_\nu(X) = \int_{\text{U}(N_f)} dU (\det U)^\nu \exp[\text{Re tr}(X^\dagger U I U^T)], \quad (7.17)$$

in terms of which we have

$$Z_\nu^{\text{eff}}(J_L, J_R) = g_\nu(\tilde{J}_L^\dagger) g_\nu(\tilde{J}_R). \quad (7.18)$$

The expansion of the integral (7.17) in powers of  $X$  was computed in [87]. Adapting eq. (40) of that reference to our case, we have with  $\alpha = N_f + \nu - 3$

$$g_\nu(X) = \mathcal{N} (\text{Pf } X)^\nu \left[ 1 + A_\alpha \text{tr } X^\dagger X + B_\alpha (\text{tr } X^\dagger X)^2 - C_\alpha \text{tr}(X^\dagger X)^2 + O(X^6) \right] \quad (7.19)$$

with coefficients

$$A_\alpha = \frac{1}{2(\alpha+2)}, \quad B_\alpha = \frac{\alpha+1}{8\alpha(\alpha+2)(\alpha+3)}, \quad C_\alpha = \frac{1}{4\alpha(\alpha+2)(\alpha+3)} \quad (7.20)$$

and a normalization factor  $\mathcal{N}$  that depends on  $N_f$  and  $\nu$  but is not important for our present purposes. ( $\mathcal{N}$  can be determined from [88] if desired, and for  $\nu = 0$  we have  $\mathcal{N} = 1$  for all  $N_f$ .) Equations (7.18) and (7.19) imply that in the limit  $J_{L/R} \rightarrow 0$  only the  $\nu = 0$  sector survives. This is analogous to QCD at  $\mu = 0$ , where in the chiral limit the topological susceptibility vanishes [89]. We thus obtain

$$\begin{aligned} \frac{Z_\nu^{\text{eff}}(J_L, J_R)}{\mathcal{N}^2 [\text{Pf}(\tilde{J}_L^\dagger) \text{Pf}(\tilde{J}_R)]^\nu} &= 1 + [A_\alpha \text{tr } \tilde{J}_L^\dagger \tilde{J}_L + B_\alpha (\text{tr } \tilde{J}_L^\dagger \tilde{J}_L)^2 - C_\alpha \text{tr}(\tilde{J}_L^\dagger \tilde{J}_L)^2 + (L \leftrightarrow R)] \\ &\quad + A_\alpha^2 (\text{tr } \tilde{J}_L^\dagger \tilde{J}_L) (\text{tr } \tilde{J}_R^\dagger \tilde{J}_R) + O(\tilde{J}^6). \end{aligned} \quad (7.21)$$

We can now match the right-hand sides of (7.9) and (7.21) to obtain sum rules for the inverse singular values,

$$\left\langle \sum'_n \frac{1}{\xi_{Ln}^2} \right\rangle_\nu = \left\langle \sum'_n \frac{1}{\xi_{Rn}^2} \right\rangle_\nu = 2\tilde{A}_\alpha, \quad \left\langle \left( \sum'_n \frac{1}{\xi_{Ln}^2} \right)^2 \right\rangle_\nu = \left\langle \left( \sum'_n \frac{1}{\xi_{Rn}^2} \right)^2 \right\rangle_\nu = 8\tilde{B}_\alpha, \quad (7.22a)$$

$$\left\langle \sum'_n \frac{1}{\xi_{Ln}^4} \right\rangle_\nu = \left\langle \sum'_n \frac{1}{\xi_{Rn}^4} \right\rangle_\nu = 4\tilde{C}_\alpha, \quad \left\langle \left( \sum'_m \frac{1}{\xi_{Rm}^2} \right) \left( \sum'_n \frac{1}{\xi_{Ln}^2} \right) \right\rangle_\nu = 4\tilde{A}_\alpha^2, \quad (7.22b)$$

where  $\tilde{A}_\alpha = (V_4\Phi_{\mathbf{I}})^2 A_\alpha$ ,  $\tilde{B}_\alpha = (V_4\Phi_{\mathbf{I}})^4 B_\alpha$ , and  $\tilde{C}_\alpha = (V_4\Phi_{\mathbf{I}})^4 C_\alpha$ . These sum rules imply

$$\left\langle \sum'_n \frac{1}{\xi_n^2} \right\rangle_\nu = 4\tilde{A}_\alpha, \quad \left\langle \left( \sum'_n \frac{1}{\xi_n^2} \right)^2 \right\rangle_\nu = 8\tilde{A}_\alpha^2 + 16\tilde{B}_\alpha, \quad \left\langle \sum'_n \frac{1}{\xi_n^4} \right\rangle_\nu = 8\tilde{C}_\alpha. \quad (7.23)$$

Note that the conditions

$$\left\langle \left( \sum'_n \frac{1}{\xi_n^2} \right)^2 \right\rangle_\nu \geq \left\langle \sum'_n \frac{1}{\xi_n^2} \right\rangle_\nu^2 \quad \text{and} \quad \left\langle \left( \sum'_n \frac{1}{\xi_{Ln}^2} - \sum'_n \frac{1}{\xi_{Rn}^2} \right)^2 \right\rangle_\nu = 16\tilde{B}_\alpha - 8\tilde{A}_\alpha^2 \geq 0 \quad (7.24)$$

both lead to the inequality  $2B_\alpha \geq A_\alpha^2$ , which is satisfied nontrivially for all  $N_f \geq 4$  since

$$2B_\alpha - A_\alpha^2 = \frac{1}{2\alpha(\alpha+2)^2(\alpha+3)} > 0. \quad (7.25)$$

The second condition in (7.24) also makes it clear that in general the left- and right-handed sums are different (for a fixed gauge field). Higher-order sum rules can be computed by expanding the partition functions to higher order in the diquark sources.

Let us now consider  $N_f = 2$  and for simplicity take  $j_R$  and  $j_L$  real. In that case (7.9) becomes

$$\begin{aligned} \frac{Z_\nu(j_L, j_R)}{(-j_L j_R)^\nu} &= 1 + j_L^2 j_R^2 \left\langle \left( \sum'_n \frac{1}{\xi_{Ln}^2} \right) \left( \sum'_m \frac{1}{\xi_{Rm}^2} \right) \right\rangle_\nu \\ &\quad + \left[ j_L^2 \left\langle \sum'_n \frac{1}{\xi_{Ln}^2} \right\rangle_\nu + j_L^4 \left\langle \sum'_{m < n} \frac{1}{\xi_{Lm}^2 \xi_{Ln}^2} \right\rangle_\nu + (L \leftrightarrow R) \right] + O(j^6). \end{aligned} \quad (7.26)$$

The finite-volume partition function obtained from (5.28) in the  $\varepsilon$ -regime is

$$Z^{\text{eff}}(j_L, j_R) = \int_{\text{U}(1)} dV \exp \left[ -2(\tilde{j}_L - \tilde{j}_R) \text{Re } V \right] \quad (7.27)$$

with  $\tilde{j}_i = j_i V_4 \Phi_{\mathbf{I}}$ . Introducing a  $\theta$ -angle as before we thus have

$$\begin{aligned} Z_\nu^{\text{eff}}(j_L, j_R) &= \int dV \frac{d\theta}{2\pi} \exp \left[ -i\nu\theta - 2 \text{Re}(\tilde{j}_L e^{-i\theta/2} - \tilde{j}_R e^{i\theta/2}) V \right] \\ &= \int_{\text{U}(1)} dL L^\nu \exp \left[ -2\tilde{j}_L \text{Re } L \right] \int_{\text{U}(1)} dR (R^\dagger)^\nu \exp \left[ 2\tilde{j}_R \text{Re } R \right] \\ &= I_\nu(-2\tilde{j}_L) I_\nu(2\tilde{j}_R), \end{aligned} \quad (7.28)$$

in analogy with the steps that led to (7.13). In the last line we have recognized the integral representation of the modified Bessel function  $I_\nu$ . Expanding in the diquark sources gives

$$\frac{Z_\nu^{\text{eff}}(j_L, j_R)}{(\nu!)^{-2}(-\tilde{j}_L \tilde{j}_R)^\nu} = 1 + \frac{\tilde{j}_L^2 + \tilde{j}_R^2}{\nu + 1} + \frac{\tilde{j}_L^2 \tilde{j}_R^2}{(\nu + 1)^2} + \frac{\tilde{j}_L^4 + \tilde{j}_R^4}{2(\nu + 1)(\nu + 2)} + O(\tilde{j}^6), \quad (7.29)$$

and matching (7.26) and (7.29) yields the sum rules

$$\left\langle \sum'_n \frac{1}{\xi_{Rn}^2} \right\rangle_\nu = \left\langle \sum'_n \frac{1}{\xi_{Ln}^2} \right\rangle_\nu = \frac{(V_4 \Phi_{\mathbf{I}})^2}{\nu + 1}, \quad \left\langle \left( \sum'_m \frac{1}{\xi_{Rm}^2} \right) \left( \sum'_n \frac{1}{\xi_{Ln}^2} \right) \right\rangle_\nu = \frac{(V_4 \Phi_{\mathbf{I}})^4}{(\nu + 1)^2}, \quad (7.30a)$$

$$\left\langle \sum'_{m < n} \frac{1}{\xi_{Rm}^2 \xi_{Rn}^2} \right\rangle_\nu = \left\langle \sum'_{m < n} \frac{1}{\xi_{Lm}^2 \xi_{Ln}^2} \right\rangle_\nu = \frac{(V_4 \Phi_{\mathbf{I}})^4}{2(\nu + 1)(\nu + 2)}. \quad (7.30b)$$

Note that the sum rules in (7.30a) follow from those in (7.22) by setting  $N_f = 2$ .

We observe that there is a “decoupling rule” for  $N_f \geq 2$ ,

$$\left\langle \left( \sum'_m \frac{1}{\xi_{Rm}^2} \right) \left( \sum'_n \frac{1}{\xi_{Ln}^2} \right) \right\rangle_\nu = \left\langle \sum'_n \frac{1}{\xi_{Rn}^2} \right\rangle_\nu \left\langle \sum'_n \frac{1}{\xi_{Ln}^2} \right\rangle_\nu. \quad (7.31)$$

A similar decoupling of the left- and right-handed modes is expected to hold for higher moments as well since the finite-volume partition function factorizes, see (7.18). It is therefore sufficient to consider only one of the factors. Considering  $N_f \geq 4$  for the time being, we divide (7.7) and (7.18) by  $[\text{Pf } J_L^\dagger]^\nu$  and then take the limit  $J_L \rightarrow 0$  to obtain

$$\int_{\text{U}(N_f)} dU (\det U)^\nu \exp [V_4 \Phi_{\mathbf{I}} \text{Re tr}(J_R^\dagger U I U^T)] \propto [\text{Pf } J_R]^\nu \left\langle \prod'_n \det^{1/2} \left( \mathbb{1} + \frac{J_R^\dagger J_R}{\xi_{Rn}^2} \right) \right\rangle_\nu \quad (7.32)$$

modulo an unimportant normalization factor. This has the same form as the mass-dependent finite-volume partition function at  $\mu = 0$  for  $N_f \geq 2$  [88],

$$Z_\nu^{\text{eff}}(\mathcal{M}) = \int_{\text{U}(2N_f)} dU (\det U)^\nu \exp \left[ \frac{1}{2} V_4 \Sigma \text{Re tr}(\mathcal{M}^\dagger U I U^T) \right] \\ \propto (\text{Pf } \mathcal{M})^\nu \left\langle \prod_{\lambda_n > 0} \det^{1/2} \left( \mathbb{1} + \frac{\mathcal{M}^\dagger \mathcal{M}}{\lambda_n^2} \right) \right\rangle_\nu \quad \text{with } \mathcal{M} = \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}, \quad (7.33)$$

where  $M$  is the quark mass matrix,  $\Sigma$  is the absolute value of the chiral condensate, the  $i\lambda_n$  are the Dirac eigenvalues at  $\mu = 0$ , and the normalization factor was omitted again. This implies the correspondence

$$\{\Phi_{\mathbf{I}}, J_R, N_f, \xi_{Rn}\}_\mu \longleftrightarrow \left\{ \frac{1}{2} \Sigma, \mathcal{M}, 2N_f, \lambda_n(> 0) \right\}_{\mu=0} \quad (7.34)$$

and similarly for  $R \leftrightarrow L$ . For  $N_f = 2$  this correspondence remains valid since the  $\text{U}(2)$ -integral in (7.33) then gives  $I_\nu(-mV_4\Sigma)$  with the quark mass  $m$ , which is to be compared with  $I_\nu(2j_R V_4 \Phi_{\mathbf{I}})$  in (7.28).

It is well known that to lowest order in the  $\varepsilon$ -regime the system can alternatively be described by chiral random matrix theory. For our present case, i.e., two-color QCD at intermediate density in the chiral limit and in the presence of diquark sources, we obtain the chiral random matrix theory<sup>44</sup>

$$Z_\nu^{\text{RMT}}(\hat{J}_L, \hat{J}_R) = g_\nu^{\text{RMT}}(\hat{J}_L^\dagger) g_\nu^{\text{RMT}}(\hat{J}_R) \quad (7.35)$$

<sup>44</sup>Random matrix theories for singular values are also considered in a very different context in [90, 91].

with

$$\begin{aligned}
g_\nu^{\text{RMT}}(\hat{J}) &= \int dA \, e^{-N \text{tr}(A^T A)} \text{Pf} \begin{pmatrix} \hat{J}^\dagger \otimes \mathbb{1}_N & \mathbb{1}_{N_f} \otimes A \\ \mathbb{1}_{N_f} \otimes (-A^T) & \hat{J} \otimes \mathbb{1}_{N+\nu} \end{pmatrix} \\
&= \int dA \, e^{-N \text{tr}(A^T A)} \prod_{f=1}^{N_f/2} \hat{j}_f^\nu \det(AA^T + |\hat{j}_f|^2), \tag{7.36}
\end{aligned}$$

where  $A$  is a real matrix of dimension  $N \times (N + \nu)$ . In the second line, we have applied an orthogonal transformation  $O$  to bring the antisymmetric matrix  $\hat{J}$  to the standard form

$$\hat{J} = O \begin{pmatrix} 0 & \text{diag}(\hat{j}_1, \dots, \hat{j}_{N_f/2}) \\ -\text{diag}(\hat{j}_1, \dots, \hat{j}_{N_f/2}) & 0 \end{pmatrix} O^T, \tag{7.37}$$

where the  $\hat{j}_f$  can be complex. For  $\nu < 0$  we have to replace  $\hat{j}_f^\nu$  by  $(\hat{j}_f^*)^{|\nu|}$  and  $AA^T$  by  $A^T A$  in (7.36). Note that the second line of (7.36) clearly exhibits the correspondence of the random matrix theory for our present case and for two-color QCD at  $\mu = 0$  with nonzero masses and without diquark sources [51].

Going through the standard steps of converting the random matrix theory to a sigma-model we find in the limit  $N \rightarrow \infty$  that  $g_\nu^{\text{RMT}}(\hat{J}) = g_\nu(\sqrt{2N}\hat{J})$ , which shows that the random-matrix partition function is equivalent to the finite-volume partition function,<sup>45</sup> provided that the dimensionless random-matrix diquark sources and singular values (i.e., the square roots of the eigenvalues of  $A^T A$ ) are related to the physical quantities by

$$\hat{J}_i = J_i V_4 \Phi_{\mathbf{I}} / \sqrt{2N} \quad \text{and} \quad \hat{\xi}_i = \xi_i V_4 \Phi_{\mathbf{I}} / \sqrt{2N} \quad (i = L, R). \tag{7.38}$$

Note that the partition function only factorizes in the chiral limit. If we include quark masses the random-matrix partition function is

$$Z_\nu^{\text{RMT}}(\hat{J}_L, \hat{J}_R, \hat{M}) = \int dA_L dA_R \, e^{-N \text{tr}(A_L^T A_L + A_R^T A_R)} \text{Pf} \begin{pmatrix} \hat{J}_L & A_L & -\hat{M}^T & 0 \\ -A_L^T & \hat{J}_L^\dagger & 0 & -\hat{M}^\dagger \\ \hat{M} & 0 & -\hat{J}_R^\dagger & -A_R \\ 0 & \hat{M}^* & A_R^T & -\hat{J}_R \end{pmatrix}, \tag{7.39}$$

where  $A_L$  and  $A_R$  are again real  $N \times (N + \nu)$  matrices. This is a natural extension of the random matrix theory constructed at high density in the absence of diquark sources [16, 93]. How the dimensionless random-matrix quark masses are related to the dimensionful masses depends on the density. At high density we have  $\hat{M} = M\sqrt{3V_4/\bar{N}}\Delta/2\pi$  [16]. What sets the scale for the masses at intermediate density is a dynamical question that cannot be answered with the methods we employ here.

For  $\mu = 0$ , the microscopic spectral correlations of the Dirac eigenvalues have been computed in chiral random matrix theory [94, 95]. The microscopic scale is defined by  $z = \lambda V_4 \Sigma$ , and as an example we quote the microscopic spectral density [96]

$$\rho_s^{N_f, \nu}(z) = \frac{z}{2} [J_a^2(z) - J_{a+1}(z)J_{a-1}(z)] + \frac{1}{2} J_a(z) \left[ 1 - \int_0^z dw J_a(w) \right], \tag{7.40}$$

<sup>45</sup>This equivalence is expected to extend to the microscopic correlations of the singular values, although this still needs to be proven using the partially quenched theory, similar to [92].

where  $a = 2N_f + |\nu|$  and  $J$  denotes the Bessel function. According to the correspondence (7.34), the microscopic density of the Dirac singular values for a given chirality,

$$\hat{\rho}_{\text{sv}}^{N_f, \nu}(x) = \lim_{V_4 \rightarrow \infty} \left\langle \sum_n \delta(x - x_n) \right\rangle_\nu \quad \text{with} \quad x_n = \xi_{Rn} V_4 \Phi_{\mathbf{I}} \text{ or } \xi_{Ln} V_4 \Phi_{\mathbf{I}}, \quad (7.41)$$

is given by

$$\hat{\rho}_{\text{sv}}^{N_f, \nu}(x) = 2\rho_s^{\frac{1}{2}N_f, \nu}(2x) \quad (7.42)$$

for  $N_f \geq 2$ , and analogously for higher-order correlation functions. By construction all sum rules derived in this section are moments of suitable microscopic correlation functions. We have checked numerically that this is indeed true for the moments of the microscopic density of the singular values.

Note that for  $N_f = 2$  we could not obtain sum rules for, e.g.,  $\langle \sum_n 1/\xi_{Rn}^4 \rangle_\nu$  or  $\langle \sum_n 1/\xi_{Ln}^4 \rangle_\nu$  from the expansion of the partition function. However, we can obtain them as moments of the microscopic density. It follows from the Taylor expansion of (7.40) that the above two sums diverge for  $\nu = 0$  and 1, which is consistent with the observation that formally setting  $N_f = 2$  in (7.22b) gives a negative or infinite result for  $\nu = 0$  or 1, respectively. But for  $\nu \geq 2$  these sums converge, and the result is identical to the one in (7.22b) with  $N_f = 2$ . Since in the random matrix theory there is nothing special about  $N_f = 2$  we expect that all sum rules derived for  $N_f \geq 4$  remain valid for  $N_f = 2$ , if convergent.

### 7.3 High density

At asymptotically high density topology is suppressed and  $m_{\text{inst}} = 0$ . In that case we expect to obtain the  $\nu = 0$  subset of the results of the preceding subsection. However, as long as  $m_{\text{inst}}$  is still nonzero the  $\nu \neq 0$  sectors are not completely suppressed. These expectations will be confirmed below.

On the QCD side nothing changes, i.e., we match to (7.9). On the low-energy effective theory side, let us again start with  $N_f \geq 4$ . Neglecting the kinetic terms in (5.17) gives

$$Z_{\nu}^{\text{eff}}(J_L, J_R) = \frac{1}{I_0(\kappa)} \int d\Sigma_L d\Sigma_R dL dR \exp \left[ \text{Re tr}(\tilde{J}_L L \Sigma_L - \tilde{J}_R R \Sigma_R) + \kappa \text{Re}(L^\dagger R)^{N_f/2} \right] \quad (7.43)$$

with  $\tilde{J}_i = J_i V_4 \Phi_{\mathbf{H}}$  and  $\kappa = 2V_4 \tilde{f}_0^2 m_{\text{inst}}^2 / N_f \geq 0$ .<sup>46</sup> The integrals over  $L, R$  are over  $U(1)$ , while those over  $\Sigma_{L,R}$  are over  $SU(N_f)/\text{Sp}(N_f)$ . The normalization factor  $1/I_0(\kappa)$  has been added to ensure  $Z_{\text{eff}}(0, 0) = 1$  as in (7.10). Introducing a  $\theta$ -angle as before and projecting onto topological sectors using (7.12) we obtain

$$\begin{aligned} Z_{\nu}^{\text{eff}}(J_L, J_R) &= \frac{1}{I_0(\kappa)} \int d\Sigma_L d\Sigma_R dL dR \frac{d\theta}{2\pi} \\ &\quad \times \exp \left[ -i\nu\theta + \text{Re tr}(\tilde{J}_L e^{-i\theta/N_f} L \Sigma_L - \tilde{J}_R e^{i\theta/N_f} R \Sigma_R) + \kappa \text{Re}(L^\dagger R)^{N_f/2} \right] \\ &= \int d\Sigma_L d\Sigma_R dL dR (LR^\dagger)^{\frac{1}{2}N_f \nu} \exp \left[ \text{Re tr}(\tilde{J}_L L \Sigma_L - \tilde{J}_R R \Sigma_R) \right] \end{aligned}$$

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<sup>46</sup>Although  $\tilde{f}_0 \sim \mu$  we have  $\kappa \rightarrow 0$  for  $\mu \rightarrow \infty$  since  $m_{\text{inst}}^2$  goes to zero much faster than  $1/\mu^2$  [67].



$$\begin{aligned}
& \times \frac{1}{I_0(\kappa)} \int \frac{d\theta}{2\pi} \exp(-i\nu\theta + \kappa \cos \theta) \\
& = [Z_\nu^{\text{eff}}(J_L, J_R) \text{ of theory } \mathbf{I}]_{\Phi_{\mathbf{I}} \rightarrow \Phi_{\mathbf{H}}} \times \frac{I_\nu(\kappa)}{I_0(\kappa)}, \tag{7.44}
\end{aligned}$$

where in the second line we have first redefined  $L \rightarrow L e^{i\theta/N_f}$ ,  $R \rightarrow R e^{-i\theta/N_f}$  and then  $e^{-2i\theta/N_f} \rightarrow e^{-2i\theta/N_f} L R^\dagger$ , and in the last line we have compared with (7.13). From (7.44) we can draw several conclusions. First, all Leutwyler-Smilga-type sum rules in theory  $\mathbf{H}$  are identical to those of theory  $\mathbf{I}$  (except for the replacement  $\Phi_{\mathbf{I}} \rightarrow \Phi_{\mathbf{H}}$ ) since the relative factor  $I_\nu(\kappa)/I_0(\kappa)$  is independent of  $J_{L/R}$  and therefore drops out when computing expectation values in sectors of fixed topology. Second, in the limit  $\kappa \rightarrow \infty$  the relative factor goes to 1 for any  $\nu$ , while for finite  $\kappa$  the sectors with  $\nu \neq 0$  are suppressed and disappear completely for  $\kappa \rightarrow 0$  as expected. Third, even in the presence of the anomaly the finite-volume partition function still factorizes into a left- and a right-handed part as in (7.18).

For  $N_f = 2$  the argument goes through in exactly the same way.

Since the sum rules for theories  $\mathbf{I}$  and  $\mathbf{H}$  are identical, it is natural to expect that the random matrix theory and the microscopic correlation functions of the singular values for  $\mathbf{H}$  are identical to those of  $\mathbf{I}$ , with  $\Phi_{\mathbf{I}} \leftrightarrow \Phi_{\mathbf{H}}$  (again, this would have to be proven using the partially quenched theory). The only difference is in the summation over topological charge. If, e.g., one wants to sum the microscopic spectral density over all sectors, one needs to take into account the  $\nu$ -dependent relative factor in (7.44), i.e.,

$$\hat{\rho}_{\text{sv}}^{\mathbf{I}}(x, \theta) = \frac{\sum_\nu \hat{\rho}_{\text{sv}}^\nu(x) e^{i\nu\theta} Z_\nu^{\mathbf{I}}}{\sum_\nu e^{i\nu\theta} Z_\nu^{\mathbf{I}}} \quad \text{and} \quad \hat{\rho}_{\text{sv}}^{\mathbf{H}}(x, \theta) = \frac{\sum_\nu \hat{\rho}_{\text{sv}}^\nu(x) e^{i\nu\theta} Z_\nu^{\mathbf{I}} I_\nu(\kappa)}{\sum_\nu e^{i\nu\theta} Z_\nu^{\mathbf{I}} I_\nu(\kappa)}, \tag{7.45}$$

where we have suppressed the arguments  $J_L$  and  $J_R$  in  $Z_\nu^{\mathbf{I}}$  and  $\hat{\rho}_{\text{sv}}$ . Note that in the limit of zero diquark sources  $Z_\nu^{\mathbf{I}} = 0$  for  $\nu \neq 0$  so that  $\hat{\rho}_{\text{sv}}(x, \theta) = \rho_{\text{sv}}^0(x)$  for both  $\mathbf{I}$  and  $\mathbf{H}$ .

#### 7.4 Low density

On the low-energy effective theory side, we use (5.4) and for simplicity take  $J_R = -J_L = jI$  with real  $j$ . Neglecting the kinetic terms we obtain the finite-volume partition function

$$Z^{\text{eff}}(\mu, j) = \int d\Sigma \exp \left[ \mu^2 F^2 V_4 \text{tr}(\Sigma B \Sigma^\dagger B) + j V_4 \Phi_{\mathbf{L}} \text{Re tr}(e^{-i\theta/N_f} \Sigma_d \Sigma) \right], \tag{7.46}$$

where the integration is over  $\text{SU}(2N_f)/\text{Sp}(2N_f)$ , the  $B^2$ -term has been absorbed in the normalization, and a  $\theta$ -angle has been introduced by  $J_R \rightarrow J_R e^{i\theta/N_f}$  and  $J_L \rightarrow J_L e^{-i\theta/N_f}$ . The expansion of (7.46) in powers of  $j$  for arbitrary  $N_f$  is formally possible, but the analytical calculation of the expansion coefficients is a challenging mathematical problem which we do not address here. Instead, we will obtain partial results for the special case  $N_f = 2$ . Because of the group isomorphisms  $\text{SU}(4) \simeq \text{SO}(6)$  and  $\text{Sp}(4) \simeq \text{SO}(5)$  we can regard the coset as  $\text{SO}(6)/\text{SO}(5) \simeq S^5$  [88]. This approach has been explored in detail [71], and in the following we use the formulation of that reference. Adapting the

unnumbered equation just before eq. (7) of [71] to our case, we have modulo a  $j$ -independent normalization factor

$$Z^{\text{eff}}(\mu, j) = \int_{S^5} d\vec{n} \exp \left[ z(n_2^2 + n_4^2) + 2wn_4 \cos \frac{\theta}{2} \right] \quad (7.47)$$

with

$$z = 8\mu^2 F^2 V_4 \quad \text{and} \quad w = 2jV_4\Phi_{\mathbf{L}}. \quad (7.48)$$

Projecting onto fixed topology using (7.12) we obtain

$$\begin{aligned} Z_\nu^{\text{eff}}(\mu, j) &= \int_{S^5} d\vec{n} e^{z(n_2^2 + n_4^2)} \int_{U(1)} d\theta e^{-i\nu\theta + 2wn_4 \cos \frac{\theta}{2}} \\ &= \int_{S^5} d\vec{n} e^{z(n_2^2 + n_4^2)} I_{2\nu}(2wn_4) \\ &= \int_{S^5} d\vec{n} e^{z(n_2^2 + n_4^2)} \frac{(wn_4)^{2\nu}}{(2\nu)!} \left[ 1 + \frac{(wn_4)^2}{2\nu + 1} + O(j^4) \right]. \end{aligned} \quad (7.49)$$

As before, we always assume  $\nu \geq 0$ . For  $\nu < 0$  we have to replace  $\nu \rightarrow |\nu|$ .

On the QCD side we again use (7.9), which now becomes

$$Z_\nu(\mu, j) \propto j^{2\nu} \left[ 1 + j^2 \left\langle \sum_n' \frac{1}{\xi_n^2} \right\rangle_\nu + O(j^4) \right], \quad (7.50)$$

where we omitted a  $j$ -independent normalization factor. Comparing the ratio of the  $j^{2\nu}$  and  $j^{2\nu+2}$  terms in (7.49) and (7.50) we obtain the sum rule

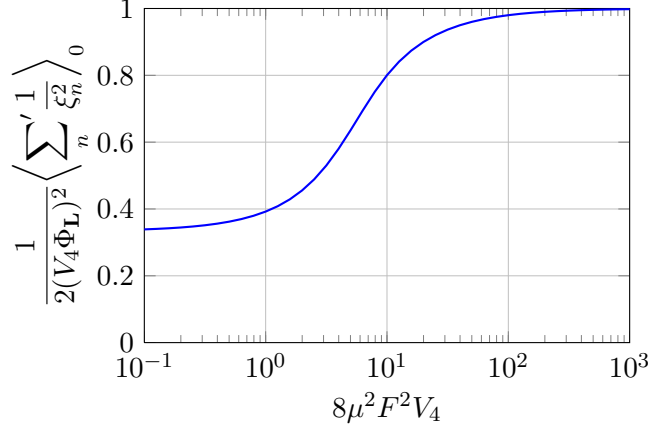
$$\left\langle \sum_n' \frac{1}{\xi_n^2} \right\rangle_\nu = \frac{4}{2\nu + 1} (V_4\Phi_{\mathbf{L}})^2 \frac{\int_{S^5} d\vec{n} e^{z(n_2^2 + n_4^2)} n_4^{2\nu+2}}{\int_{S^5} d\vec{n} e^{z(n_2^2 + n_4^2)} n_4^{2\nu}}. \quad (7.51)$$

For  $z = 0$ , (7.51) reproduces the sum rule in [88], as it should.<sup>47</sup> In the opposite limit  $z \rightarrow \infty$ , the r.h.s. converges to  $2(V_4\Phi_{\mathbf{L}})^2/(\nu+1)$  in agreement with (7.30a) (with  $\Phi_{\mathbf{I}} \leftrightarrow \Phi_{\mathbf{L}}$ ). The analytical calculation of the r.h.s. for finite  $z$  is nontrivial, and we only consider the special case  $\nu = 0$ , for which we obtain

$$\left\langle \sum_n' \frac{1}{\xi_n^2} \right\rangle_0 = 2(V_4\Phi_{\mathbf{L}})^2 \left( \frac{z}{e^z - z - 1} + 1 - \frac{2}{z} \right), \quad (7.52)$$

the plot of which is shown as a function of  $z$  in figure 7. The proof of this result is given in appendix G. The sum rules for the right- and left-handed singular values are also given by (7.52) but with the r.h.s. divided by 2. Higher-order sum rules can be obtained as usual by expanding (7.49) and (7.50) to higher orders in  $j$ .

<sup>47</sup>In this comparison we need to be careful. First, our  $\Phi_{\mathbf{L}}$  corresponds to  $\Sigma/2$  in [88]. Second, all singular values are doubly degenerate (for  $\beta = 1$ ) at  $\mu = 0$ , which means that the sum rule in [88] should be compared to our result divided by 2.



**Figure 7.**  $\mu$ -dependence of the spectral sum rule (7.52) in theory **L** in the sector  $\nu = 0$ . The curve converges to  $1/3$  as  $\mu \rightarrow 0$  and to  $1$  as  $\mu \rightarrow \infty$ .

As before, to lowest order in the  $\varepsilon$ -regime we can also describe the system by a random matrix theory, which at low density we can obtain by adding diquark sources to the known two-matrix model [18, 97], resulting in

$$Z_\nu^{\text{RMT}}(\hat{\mu}, \hat{J}_L, \hat{J}_R, \hat{M}) = \int dC dD e^{-2N \text{tr}(C^T C + D^T D)} \text{Pf} \begin{pmatrix} \hat{J}_L & C - \hat{\mu}D & -\hat{M}^T & 0 \\ -C^T + \hat{\mu}D^T & \hat{J}_L^\dagger & 0 & -\hat{M}^\dagger \\ \hat{M} & 0 & -\hat{J}_R^\dagger & -C - \hat{\mu}D \\ 0 & \hat{M}^* & C^T + \hat{\mu}D^T & -\hat{J}_R \end{pmatrix}. \quad (7.53)$$

For earlier approaches, see [98, 99]. In (7.53),  $C$  and  $D$  are again real  $N \times (N + \nu)$  matrices. Converting (7.53) to a sigma-model we find in the limit  $N \rightarrow \infty$  that the random matrix parameters are related to physical quantities by

$$\hat{\mu}^2 = 2\mu^2 F^2 V_4 / N, \quad \hat{M} = M V_4 \Phi_{\mathbf{L}} / 2N, \quad \hat{J}_i = J_i V_4 \Phi_{\mathbf{L}} / 2N, \quad \hat{\xi}_i = \xi_i V_4 \Phi_{\mathbf{L}} / 2N, \quad (7.54)$$

where in the chiral limit the  $\xi_L$  ( $\xi_R$ ) are the singular values of  $C - \hat{\mu}D$  ( $C + \hat{\mu}D$ ). Note that for  $\hat{\mu} \neq 1$  the partition function does not factorize even in the chiral limit. However, for  $\hat{\mu} = 1$  (“maximum non-Hermiticity”) (7.53) reduces to (7.39), and if the chiral limit is taken it factorizes again. The computation of the microscopic correlation functions of the eigenvalues and/or singular values from this random matrix theory is a complicated mathematical task which we do not attempt here.

In section 7.1 we have seen that the  $\varepsilon$ -regimes of the effective theories **L** and **I** do not overlap. Nevertheless, as just noted, the random matrix theory (7.39) for **I** is the  $\hat{\mu} \rightarrow 1$  limit of the random matrix theory (7.53) for **L**. This is consistent with the observation that the sum of the sum rules for  $1/\xi_{Rn}^2$  and  $1/\xi_{Ln}^2$  in (7.30a) is the  $z \rightarrow \infty$  limit of (7.51) for all  $\nu$  (with  $\Phi_{\mathbf{I}} \leftrightarrow \Phi_{\mathbf{L}}$ ). We are therefore tempted to conjecture that all sum rules of **I** are the  $z \rightarrow \infty$  limits of the sum rules for **L**.

## 8 Conclusions and outlook

In this paper, we have studied the singular values of the Dirac operator in QCD-like theories with Dyson indices  $\beta = 1, 2$ , and  $4$  at nonzero chemical potential  $\mu$ . We pointed out that the Dirac singular values are always real and nonnegative and that the scale of the singular value spectrum is set by the diquark condensate at any  $\mu$ . This is in contrast to the Dirac eigenvalues, which spread into the complex plane at nonzero  $\mu$  and whose scale is set by the chiral condensate at small  $\mu$  [6] and by the BCS gap at large  $\mu$  [15, 16, 18]. We derived Banks-Casher-type relations for all three values of  $\beta$  and then concentrated on the  $\beta = 1$  case, for which we identified three different low-energy effective theories with diquark sources at low, intermediate, and high density within the whole BEC-BCS crossover region, and clarified how they are related to each other from the point of view of integrating out heavy degrees of freedom. We derived exact results, such as Smilga-Stern-type relations and Leutwyler-Smilga-type sum rules, which (together with the Banks-Casher-type relation) concern the connection between the singular value spectrum and diquark condensation. We have also identified the  $\varepsilon$ -regimes of the effective theories and constructed the corresponding chiral random matrix theories, from which the microscopic spectral correlation functions of the singular values can be determined. Our results can in principle be tested in future lattice QCD simulations. This should provide a value of the diquark condensate at any density, by which the conjectured BEC-BCS crossover could be confirmed numerically.

It is evident from our results that the existence of a nonzero diquark condensate at any chemical potential implies an accumulation of the Dirac singular values at the origin at any quark density. In the case of the QCD vacuum, near-zero Dirac eigenvalues responsible for chiral symmetry breaking are believed to originate from instantons, as illustrated by the instanton liquid model [100]. One might thus naively expect that the accumulation of near-zero singular values in dense QCD is also attributable to instantons. However, this is not the case. Although instanton effects are presumably important at small and intermediate density, they are suppressed at sufficiently high density where the one-gluon exchange interaction is more important for the formation of diquark pairing. The presence of the Fermi surface is crucial in this mechanism.

Let us discuss some possible extensions of the present work. First, here we have concentrated on nonzero diquark sources without quark masses, but the generalization to include quark masses looks straightforward. In that way we can study not only the diquark condensate but also the chiral condensate or the BCS gap as a function of the chemical potential. Second, the results obtained in sections 5 through 7 could also be generalized to theories with  $\beta = 2$  and  $\beta = 4$ . Third, it would be very interesting to generalize our results to the color-superconducting phases of three-color QCD. Unfortunately, this is not straightforward since the diquark source is no longer gauge invariant, though the magnitude of the diquark condensate is. The object  $D(\mu)^\dagger D(\mu)$  obviously exists in three-color QCD and can be studied, but it is currently unclear to us how its spectrum is related to physical observables. It would also be interesting to find out whether (and if so, how) the gauge invariant four-quark condensate is related to the Dirac eigenvalue or singular value spectrum in three-color QCD.

We conclude with a short discussion of a phenomenon analogous to the conjectured BEC-BCS crossover in QCD-like theories studied in this paper, i.e., the conjectured hadron-quark continuity in three-color QCD at nonzero baryon density [101]. Recently, the existence of a bound state of baryons, the H dibaryon (with mass  $m_H$ ), was observed in lattice QCD simulations at zero density [102, 103]. This implies (for three degenerate flavors) that, as we go to nonzero baryon density, first BEC of H dibaryons occurs, since due to its binding energy a bosonic H dibaryon has a smaller excitation energy  $m_H/2 - \mu_B$  (per baryon) than a baryon. In this BEC state  $U(1)_B$  and chiral symmetry are broken spontaneously, and thus this state has the same symmetry-breaking pattern as the color-flavor-locked phase at high density, where both  $U(1)_B$  and chiral symmetry are broken by the diquark condensate. Therefore these two phases can be continuously connected without any phase transition (hadron-quark continuity) as first conjectured by Schäfer and Wilczek [101]. An explicit realization of this conjecture was given within generalized Ginzburg-Landau theory [104–106] and the Nambu–Jona-Lasinio model [39], where it was shown that the QCD axial anomaly can lead to a crossover between the hadronic and the CFL phase. Although it is not yet clear what happens physically between these two regions, there could be successive changes of states, from BEC of dibaryons to BCS pairing of dibaryons to BEC of diquarks and finally to BCS pairing of diquarks.

In the real world, flavor symmetry is explicitly broken due to the heavy strange quark mass, and the existence of the H dibaryon as well as the scenario above may be modified. Actually, it is empirically believed that a nuclear liquid-gas phase transition takes place as the baryon density increases. In order to understand high-density matter in the real world, it would thus be crucial to take into account the effects of flavor symmetry breaking. Unfortunately, these effects cannot be directly studied in lattice QCD simulations even for QCD-like theories since, e.g., nondegenerate quark masses in two-color QCD give rise to the sign problem. It is an important future problem to figure out how one can study flavor symmetry breaking at nonzero density on the lattice.

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## A Definitions and conventions

Unless stated otherwise we always work in Euclidean space. The  $\gamma$ -matrices are Hermitian and satisfy  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$  ( $\mu, \nu = 1, \dots, 4$ ). We choose the chiral representation given by

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (\text{A.1})$$

where the  $\sigma_i$  ( $i = 1, 2, 3$ ) are the usual Pauli matrices. A Dirac spinor  $\psi$  can be written in terms of two Weyl spinors  $\psi_R$  and  $\psi_L$ ,

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad (\text{A.2})$$

and we have

$$\bar{\psi} = \psi^\dagger \gamma_4 = (\psi_L^\dagger \ \psi_R^\dagger). \quad (\text{A.3})$$

The projection operators on the right- and left-handed sectors are

$$P_R = \frac{1}{2}(\mathbb{1} + \gamma_5) \quad \text{and} \quad P_L = \frac{1}{2}(\mathbb{1} - \gamma_5), \quad (\text{A.4})$$

and we have, with a slight abuse of notation,

$$\begin{aligned} P_R \psi &= \psi_R, & \bar{\psi} P_R &= \psi_L^\dagger, \\ P_L \psi &= \psi_L, & \bar{\psi} P_L &= \psi_R^\dagger. \end{aligned} \quad (\text{A.5})$$

The charge conjugation matrix satisfies

$$C \gamma_\mu C^{-1} = -\gamma_\mu^T \quad \text{and} \quad C^T = -C. \quad (\text{A.6})$$

We adopt the choice  $C = i\gamma_4\gamma_2$ , for which we have

$$C^{-1} = C^\dagger = C, \quad (\text{A.7})$$

$$[C, \gamma_1] = \{C, \gamma_2\} = [C, \gamma_3] = \{C, \gamma_4\} = [C, \gamma_5] = 0. \quad (\text{A.8})$$

We also define the  $N_f$ -dimensional antisymmetric matrix

$$I = \begin{pmatrix} 0 & -\mathbb{1}_{N_f/2} \\ \mathbb{1}_{N_f/2} & 0 \end{pmatrix}. \quad (\text{A.9})$$

## B Partition functions with diquark sources

In this appendix we derive the singular value representations of the partition functions with diquark sources for the theories with  $\beta = 1, 2$ , and 4, taking exact zero modes into account and showing how the positivity of the path integral measure is determined.

### B.1 Two-color QCD ( $\beta = 1$ )

We first consider two-color QCD. We need to evaluate the Pfaffian of the operator  $W$  in (2.6) in the chiral limit, regarded as an infinite-dimensional antisymmetric matrix,

$$\text{Pf}(W) = \text{Pf} \begin{pmatrix} C\tau_2(J_R P_R + J_L P_L) & -D(\mu)^T \\ D(\mu) & -C\tau_2(J_R^\dagger P_L + J_L^\dagger P_R) \end{pmatrix}. \quad (\text{B.1})$$

Note that the transpose  $D(\mu)^T$  includes transposition of the space-time indices, in addition to the color and spinor indices. Below we will give two treatments of  $\text{Pf}(W)$ . They lead to the same form for the partition function, which underscores the correctness of the result.

### B.1.1 Rigorous derivation

To evaluate the Pfaffian it is useful to employ a specific functional basis. Here we use the eigenfunctions of  $D^\dagger D$  and  $DD^\dagger$  introduced in section 3.2,<sup>48</sup>

$$D^\dagger D \varphi_n = \xi_n^2 \varphi_n, \quad \int d^4x \varphi_m^\dagger \varphi_n = \delta_{mn}, \quad (\text{B.2a})$$

$$DD^\dagger \tilde{\varphi}_n = \xi_n^2 \tilde{\varphi}_n, \quad \int d^4x \tilde{\varphi}_m^\dagger \tilde{\varphi}_n = \delta_{mn}, \quad (\text{B.2b})$$

where for  $\xi_n > 0$  we have

$$\varphi_n = \frac{1}{\xi_n} D^\dagger \tilde{\varphi}_n \quad \text{and} \quad \tilde{\varphi}_n = \frac{1}{\xi_n} D \varphi_n. \quad (\text{B.3})$$

Then the fields  $\psi$  and  $\bar{\psi}$  in (2.4) can be expanded in the bases  $\{\varphi_n\}$  and  $\{\tilde{\varphi}_n^\dagger\}$ , respectively. We need the help of the following lemma to proceed.

**Lemma.** *Without loss of generality, we can assume*

$$\varphi_n = p_n C \tau_2 \varphi_n^*, \quad (\text{B.4a})$$

$$\tilde{\varphi}_n = -p_n C \tau_2 \tilde{\varphi}_n^* \quad (\text{B.4b})$$

for some  $p_n \in \mathbb{C}$  with  $|p_n| = 1$ .

*Proof.* For two-color QCD, it can be shown from (2.2a) that

$$(D^\dagger D)^* = C \tau_2 D^\dagger D C \tau_2, \quad (\text{B.5a})$$

$$(DD^\dagger)^* = C \tau_2 DD^\dagger C \tau_2. \quad (\text{B.5b})$$

Using these properties as well as (B.2), it follows that

$$D^\dagger D (C \tau_2 \varphi_n^*) = \xi_n^2 (C \tau_2 \varphi_n^*), \quad (\text{B.6a})$$

$$DD^\dagger (C \tau_2 \tilde{\varphi}_n^*) = \xi_n^2 (C \tau_2 \tilde{\varphi}_n^*). \quad (\text{B.6b})$$

Therefore both  $\varphi_n$  and  $C \tau_2 \varphi_n^*$  ( $\tilde{\varphi}_n$  and  $C \tau_2 \tilde{\varphi}_n^*$ ) are eigenfunctions of  $D^\dagger D$  ( $DD^\dagger$ ) with the same eigenvalue  $\xi_n^2$ . Then two possibilities arise:

1.  $\varphi_n$  and  $C \tau_2 \varphi_n^*$  are linearly independent, and the eigenvalue  $\xi_n^2$  is (at least) doubly degenerate.
2.  $\varphi_n$  and  $C \tau_2 \varphi_n^*$  are linearly dependent, and the eigenvalue  $\xi_n^2$  is not degenerate.

In the first case, we can redefine  $\mathcal{N}_1(\varphi_n + p_n C \tau_2 \varphi_n^*)$  as  $\varphi_n$  and  $\mathcal{N}_2(\varphi_n - p_n C \tau_2 \varphi_n^*)$  as  $\varphi_{n+1}$  with normalization constants  $\mathcal{N}_{1,2}$  and arbitrary  $p_n \in \text{U}(1)$  so that<sup>49</sup>

$$\varphi_n = p_n C \tau_2 \varphi_n^* \quad \text{and} \quad \varphi_{n+1} = -p_n C \tau_2 \varphi_{n+1}^* \equiv p_{n+1} C \tau_2 \varphi_{n+1}^*, \quad (\text{B.7})$$

<sup>48</sup>Note that these eigenfunctions are ordinary  $c$ -number functions and not Grassmannian.

<sup>49</sup>This redefinition does not change the chirality because  $C \tau_2$  commutes with  $\gamma_5$ .

where we used  $(C\tau_2)^* = C\tau_2$  and  $(C\tau_2)^2 = \mathbb{1}$ . In the second case, it can easily be shown that there exists a phase  $p_n \in \text{U}(1)$  such that  $\varphi_n = p_n C\tau_2 \varphi_n^*$ , so the desired relation holds automatically. This completes the proof of (B.4a). Note that we have shown this for both zero and nonzero modes.

Let us proceed to the proof of (B.4b). For nonzero modes ( $\xi_n > 0$ ), we have

$$\tilde{\varphi}_n^* = \frac{1}{\xi_n} D^* \varphi_n^* = \frac{1}{\xi_n} (-C\tau_2 D C\tau_2) p_n^* C\tau_2 \varphi_n = -p_n^* C\tau_2 \frac{1}{\xi_n} D \varphi_n = -p_n^* C\tau_2 \tilde{\varphi}_n. \quad (\text{B.8})$$

Multiplying both sides by  $-p_n C\tau_2$  we obtain (B.4b). Finally we consider the case of zero modes ( $\xi_n = 0$ ). Although there is no obvious relation between the zero modes of  $D$  and those of  $D^\dagger$ , their numbers are equal, see (D.24). Choosing the phases of the zero modes of  $DD^\dagger$  one can make (B.4b) hold.  $\square$

Now it is straightforward to obtain the matrix elements of  $W$  in the bases  $\{\varphi_n\}$  (for  $\psi$ ) and  $\{\tilde{\varphi}_n^\dagger\}$  (for  $\bar{\psi}$ ). We consider separately the four blocks of  $W$ .

- (1, 1)-block:

$$\begin{aligned} \int d^4x \varphi_m^T [C\tau_2 (J_R P_R + J_L P_L)] \varphi_n &= \int d^4x p_m \varphi_m^\dagger (J_R P_R + J_L P_L) \varphi_n \\ &= p_n \delta_{mn} J_{R/L}, \end{aligned} \quad (\text{B.9})$$

where in the first step we have used (B.4a) and in the second step  $R/L$  corresponds to the handedness of  $\varphi_n$ .

- (2, 1)-block:

$$\int d^4x \tilde{\varphi}_m^\dagger D \varphi_n = \xi_n \delta_{mn}. \quad (\text{B.10})$$

- (1, 2)-block:

$$\int d^4x \varphi_m^T (-D^T) \tilde{\varphi}_n^* = -\xi_n \delta_{mn}. \quad (\text{B.11})$$

- (2, 2)-block:

$$\begin{aligned} \int d^4x \tilde{\varphi}_m^\dagger [-C\tau_2 (J_R^\dagger P_L + J_L^\dagger P_R)] \tilde{\varphi}_n^* &= \int d^4x \tilde{\varphi}_m^\dagger (J_R^\dagger P_L + J_L^\dagger P_R) p_n^* \tilde{\varphi}_n \\ &= p_n^* \delta_{mn} J_{R/L}^\dagger, \end{aligned} \quad (\text{B.12})$$

where in the first step we have used (B.4b) and in the second step  $R/L$  corresponds to the *opposite* of the handedness of  $\tilde{\varphi}_n$ .

Collecting all results, we obtain

$$\text{Pf}(W) = \text{Pf} \begin{pmatrix} \text{diag}(p_n) \otimes J_{R/L} & -\text{diag}(\xi_n) \otimes \mathbb{1}_{N_f} \\ \text{diag}(\xi_n) \otimes \mathbb{1}_{N_f} & \text{diag}(p_n^*) \otimes J_{R/L}^\dagger \end{pmatrix} = \prod_n \text{Pf} \begin{pmatrix} J_{R/L} & -\xi_n \\ \xi_n & J_{R/L}^\dagger \end{pmatrix}, \quad (\text{B.13})$$



where we used  $p_n p_n^* = 1$ . The contribution from zero modes can be read off as

$$\prod_{n: \xi_n=0} \text{Pf} \begin{pmatrix} J_{R/L} & 0 \\ 0 & J_{R/L}^\dagger \end{pmatrix} = [\text{Pf}(J_R) \text{Pf}(J_L^\dagger)]^{n_R} [\text{Pf}(J_R^\dagger) \text{Pf}(J_L)]^{n_L}, \quad (\text{B.14})$$

where  $n_{R,L} \geq 0$  denotes the number of zero modes of each handedness.

The contribution from a nonzero mode is given by<sup>50</sup>

$$\det^{1/2} \begin{pmatrix} J_{R/L} & -\xi_n \\ \xi_n & J_{R/L}^\dagger \end{pmatrix} = \begin{cases} \det^{1/2}(\xi_n^2 + J_R^\dagger J_R) & \text{for right-handed } \varphi_n, \\ \det^{1/2}(\xi_n^2 + J_L^\dagger J_L) & \text{for left-handed } \varphi_n, \end{cases} \quad (\text{B.15})$$

where we used the fact that the handedness of  $\varphi_n$  is opposite to that of  $\tilde{\varphi}_n$  for  $\xi_n \neq 0$ . Since (B.15) is manifestly positive definite, the (non-) positivity of the measure is determined by (B.14).

Summarizing, we find that the full partition function of two-color QCD with diquark sources in the chiral limit reduces to the following expression in terms of the singular values of  $D(\mu)$ ,

$$Z(J_L, J_R) = \left\langle [\text{Pf}(J_R) \text{Pf}(J_L^\dagger)]^{n_R} [\text{Pf}(J_R^\dagger) \text{Pf}(J_L)]^{n_L} \times \prod_n' \det^{1/2}(\xi_n^2 + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L) \right\rangle_{\text{YM}}, \quad (\text{B.16})$$

where the primed product runs over all nonzero singular values. The above expression allows for accidental zero modes, but they are of measure zero, see section 3.1. Therefore generically we have  $(n_R, n_L) = (\nu, 0)$  for  $\nu \geq 0$  and  $(0, -\nu)$  for  $\nu < 0$ . Note also that  $\text{Pf}(J_R^\dagger) \text{Pf}(J_L) = [\text{Pf}(J_R) \text{Pf}(J_L^\dagger)]^*$ .

### B.1.2 Short derivation

The derivation of the singular value representation of the partition function in the last subsection is rigorous but lengthy. Here we will give a less rigorous but shorter derivation of the same expression.

Let us assume  $\mu = 0$  so that the extended flavor symmetry  $\text{SU}(2N_f)$  is intact in the absence of diquark sources. In computing  $\text{Pf}(W)$  let us separate the contributions from zero modes and those from nonzero modes. First, in the space of nonzero modes, we compute the square of the Pfaffian in (2.6), which is equal to its determinant,

$$\text{Pf}'(W)^2 = \det' \begin{pmatrix} C\tau_2(J_R P_R + J_L P_L) & C\tau_2 D^\dagger C\tau_2 \\ D & -C\tau_2(J_R^\dagger P_L + J_L^\dagger P_R) \end{pmatrix}, \quad (\text{B.17})$$

where the prime on both sides indicates the omission of zero modes and we used  $-D^T = C\tau_2 D^\dagger C\tau_2$  in the  $(1, 2)$  block. If we interchange the first and the second column, there will be a factor  $(-1)^d$ , with  $d$  the total dimension of the space spanned by the nonzero modes

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<sup>50</sup>Rigorously speaking,  $\text{Pf}$  and  $\det^{1/2}$  may differ by a sign. However, a  $\nu$ -independent multiplicative constant can safely be omitted without changing expectation values of observables.

(we assume a suitable regularization such that  $d < \infty$ ). Since all nonzero modes are paired by  $\gamma_5$ ,  $d$  is an even integer and  $(-1)^d = 1$ . Thus

$$\begin{aligned}
\text{Pf}'(W)^2 &= \det' \begin{pmatrix} C\tau_2 D^\dagger C\tau_2 & C\tau_2(J_R P_R + J_L P_L) \\ -C\tau_2(J_R^\dagger P_L + J_L^\dagger P_R) & D \end{pmatrix} \\
&= \det' \begin{pmatrix} C\tau_2 & 0 \\ 0 & C\tau_2 \end{pmatrix} \begin{pmatrix} D^\dagger C\tau_2 & J_R P_R + J_L P_L \\ -J_R^\dagger P_L - J_L^\dagger P_R & C\tau_2 D \end{pmatrix} \\
&= \det' \begin{pmatrix} D^\dagger C\tau_2 & J_R P_R + J_L P_L \\ -J_R^\dagger P_L - J_L^\dagger P_R & C\tau_2 D \end{pmatrix}. \tag{B.18}
\end{aligned}$$

Using the formula  $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - ACA^{-1}B)$  that holds when the blocks are square matrices of the same dimension and  $A$  is invertible, we obtain

$$\begin{aligned}
\text{Pf}'(W)^2 &= \det' \left( D^\dagger D + D^\dagger C\tau_2(J_R^\dagger P_L + J_L^\dagger P_R)(D^\dagger C\tau_2)^{-1}(J_R P_R + J_L P_L) \right) \\
&= \det'(D^\dagger D + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L). \tag{B.19}
\end{aligned}$$

We stress that the above formula cannot be used in the full eigenspace of  $D$  because  $D^{-1}$  does not exist if there are zero modes. Since (B.19) is positive definite we can take its square root naively to obtain a formula for  $\text{Pf}'(W)$ .

The next task is to incorporate zero modes (where from now on we ignore accidental zero modes). We do this by switching from quark masses to diquark sources. Let us recall that the mass term in (2.3) can be written as [20]

$$\bar{\psi}(MP_L + M^\dagger P_R)\psi = \frac{1}{2}\Psi^\dagger \sigma_2 \tau_2 \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} \Psi^* + \text{h.c.} \tag{B.20}$$

with the “extended” spinor  $\Psi \equiv \begin{pmatrix} \psi_R \\ \sigma_2 \tau_2 \psi_L^* \end{pmatrix}$ .<sup>51</sup> As is well known [4], the contribution of zero modes to the fermion determinant in the absence of diquark sources is  $(\det M^\dagger)^\nu$  for  $\nu \geq 0$  and  $(\det M)^{-\nu}$  for  $\nu < 0$ .<sup>52</sup> This expression is not desirable, as it does not manifestly show the spurious invariance under  $\text{SU}(2N_f)$ .<sup>53</sup> Instead, we write  $\det M$  as a Pfaffian [88, Eq. (4.13)],

$$\det M = (-1)^{N_f/2} \text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}, \tag{B.21}$$

<sup>51</sup>The symbol  $\Psi$  used here is not to be confused with  $\Psi$  defined in (2.4).

<sup>52</sup>This differs from [4, 88]. The reason seems to be that our  $\gamma_5$  differs from their  $\gamma_5$  by a minus sign. (The definitions of the mass term and of  $\nu$  in (2.11) are identical.) Thus  $\nu = n_R - n_L$  in this paper, whereas  $\nu = n_L - n_R$  in [4, 88]. One should be careful when comparing results in this paper with those in [4, 88].

<sup>53</sup>For a fixed quark mass,  $\text{SU}(2N_f)$  is broken to  $\text{U}(1)_B \times \text{SU}(N_f)_R \times \text{SU}(N_f)_L$ , but the initial symmetry is kept intact if we formally transform the quark mass as a spurion field, see (B.22).

which makes the invariance under  $U \in \text{SU}(2N_f)$  explicit, i.e.,

$$\text{Pf} \left[ U \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} U^T \right] = \det U \cdot \text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = \text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}. \quad (\text{B.22})$$

Similarly the diquark source in (2.3) can be cast in the form

$$\frac{1}{2} \psi^T C \tau_2 (J_R P_R + J_L P_L) \psi + \text{h.c.} = \frac{1}{2} \Psi^\dagger \sigma_2 \tau_2 \begin{pmatrix} -J_R^\dagger & 0 \\ 0 & J_L \end{pmatrix} \Psi^* + \text{h.c.} \quad (\text{B.23})$$

Comparing this with (B.20) we notice that the diquark source is obtained if we simply replace  $\begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}$  by  $\begin{pmatrix} -J_R^\dagger & 0 \\ 0 & J_L \end{pmatrix}$ . Therefore the contribution of zero modes (for  $\nu < 0$ ) in the chiral limit and with nonzero diquark sources can be found by the replacement<sup>54</sup>

$$\begin{aligned} (\det M)^{-\nu} &= (-1)^{\nu N_f/2} \text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}^{-\nu} \\ &\longrightarrow (-1)^{\nu N_f/2} \text{Pf} \begin{pmatrix} -J_R^\dagger & 0 \\ 0 & J_L \end{pmatrix}^{-\nu} = [\text{Pf}(J_R^\dagger) \text{Pf}(J_L)]^{-\nu}, \end{aligned} \quad (\text{B.24})$$

and similarly for  $\nu \geq 0$ . Combining the square root of (B.19) with (B.24) we finally obtain for the partition function

$$Z(J_L, J_R) = \left\langle \left\{ \begin{aligned} &[\text{Pf}(J_R) \text{Pf}(J_L^\dagger)]^\nu \\ &[\text{Pf}(J_R^\dagger) \text{Pf}(J_L)]^{-\nu} \end{aligned} \right\} \sqrt{\det'(D^\dagger D + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L)} \right\rangle_{\text{YM}}, \quad (\text{B.25})$$

where the first (second) line in curly braces applies to  $\nu \geq 0$  ( $\nu < 0$ ). Assuming that  $\mu \neq 0$  does not change the form of this expression, we arrive at (B.16) in the previous subsection. Note that this argument hinges on the assumption that the contribution of exact zero modes at  $\mu \neq 0$  is the same as at  $\mu = 0$ . We also point out that, since  $D = -D^\dagger$  at  $\mu = 0$ , we could also have written  $D^\dagger D$  appearing in (B.25) as  $-D^2$ , but this would not have generalized to  $\mu \neq 0$ .

## B.2 QCD with isospin chemical potential ( $\beta = 2$ )

Here we only sketch the shorter derivation of the singular value representation of the partition function, i.e., the extension of the argument in section B.1.2 to  $\beta = 2$ , and omit the lengthy and rigorous version.

First, working along similar lines as in section B.1.2, the contribution from nonzero modes to the partition function can be shown to take the form

$$\det'(D^\dagger D + \rho \rho^* P_R + \lambda \lambda^* P_L). \quad (\text{B.26})$$

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<sup>54</sup>A precise account of this procedure goes as follows. We start with a certain diquark source in the chiral limit. By a suitable  $\text{SU}(2N_f)$  transformation we can rotate it into the form of the mass term, for which we know that the zero modes contribute  $(\det M)^{-\nu}$  to the fermion determinant. Then we rotate inversely, bringing the mass term back to our original diquark source. This is how we get (B.24) so quickly.

Next we consider the zero modes. In order to figure out the mapping between quark masses and pionic sources, we go to a new basis

$$\Psi_R \equiv P_R \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \quad \text{and} \quad \Psi_L \equiv P_L \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}. \quad (\text{B.27})$$

These are the  $\beta = 2$  counterparts of the extended spinor  $\Psi$  introduced after (B.20) for  $\beta = 1$ . Then the mass term for  $M = \text{diag}(m_u, m_d)$  is

$$\bar{u}(m_u P_L + m_u^* P_R)u + \bar{d}(m_d P_L + m_d^* P_R)d = \bar{\Psi}_R M \Psi_L + \bar{\Psi}_L M^\dagger \Psi_R. \quad (\text{B.28})$$

The zero-mode contribution to the fermion determinant is given by  $(\det M)^{-\nu}$  for  $\nu < 0$  and  $(\det M^\dagger)^\nu$  for  $\nu \geq 0$ . This form is manifestly invariant under the spurious  $\text{SU}(2)_R \times \text{SU}(2)_L$  rotation of  $M$ . On the other hand, the pionic source term reads

$$\bar{u}(\lambda^* P_R + \rho^* P_L)d + \bar{d}(\rho P_R + \lambda P_L)u = \bar{\Psi}_R \begin{pmatrix} 0 & \rho^* \\ \lambda & 0 \end{pmatrix} \Psi_L + \bar{\Psi}_L \begin{pmatrix} 0 & \lambda^* \\ \rho & 0 \end{pmatrix} \Psi_R. \quad (\text{B.29})$$

Comparing with the mass term, we obtain the correspondence

$$M \longleftrightarrow \begin{pmatrix} 0 & \rho^* \\ \lambda & 0 \end{pmatrix} \quad \text{and} \quad M^\dagger \longleftrightarrow \begin{pmatrix} 0 & \lambda^* \\ \rho & 0 \end{pmatrix}, \quad (\text{B.30})$$

i.e., there is a mass term that can be rotated to a given pionic source term under the action of  $\text{SU}(2)_R \times \text{SU}(2)_L$ . Thus the contribution of zero modes is given by

$$\det \begin{pmatrix} 0 & \lambda^* \\ \rho & 0 \end{pmatrix}^\nu = (-\rho \lambda^*)^\nu \quad \text{for } \nu \geq 0, \quad (\text{B.31a})$$

$$\det \begin{pmatrix} 0 & \rho^* \\ \lambda & 0 \end{pmatrix}^{-\nu} = (-\rho^* \lambda)^{-\nu} \quad \text{for } \nu < 0. \quad (\text{B.31b})$$

Summarizing, we find for the partition function

$$Z(\rho, \lambda) = \left\langle \left\{ \begin{array}{l} (-\rho \lambda^*)^\nu \\ (-\rho^* \lambda)^{-\nu} \end{array} \right\} \det'(D^\dagger D + \rho \rho^* P_R + \lambda \lambda^* P_L) \right\rangle_{\text{YM}} \quad \text{for } \left\{ \begin{array}{l} \nu \geq 0 \\ \nu < 0 \end{array} \right\}. \quad (\text{B.32})$$

In deriving (B.32) we ignored accidental zero modes. The result can straightforwardly be extended to include them, and we then obtain (2.14).

### B.3 QCD with adjoint fermions ( $\beta = 4$ )

Again we only sketch the shorter and less rigorous derivation. First, we consider the contribution of nonzero modes. Along similar lines as for  $\beta = 1$  and 2 we obtain

$$\sqrt{\det'(D^\dagger D + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L)}, \quad (\text{B.33})$$

but this is not the end of the story. Using (2.16a) we can show  $(D^\dagger D)^* = C D^\dagger D C$ , from which it follows that a state  $C \varphi_n^*$  associated with an eigenstate  $\varphi_n$  of  $D^\dagger D$  is also an

eigenstate of  $D^\dagger D$  with the same eigenvalue. As  $\varphi_n$  and  $C\varphi_n^*$  are linearly independent because of  $(CK)^2 = -\mathbb{1}$ , all eigenvalues of  $D^\dagger D$  are doubly degenerate (including zero modes). This allows us to take the square of (B.33),

$$\det''(D^\dagger D + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L), \quad (\text{B.34})$$

where in  $\det''$  each degenerate eigenvalue of  $D^\dagger D$  is counted only once.

Next we consider the zero-mode contributions. With the extended spinor  $\Psi \equiv \begin{pmatrix} \psi_R \\ i\sigma_2 \psi_L^* \end{pmatrix}$  of length  $2N_f$ ,<sup>55</sup> the mass term becomes

$$\bar{\psi}(MP_L + M^\dagger P_R)\psi = -\frac{1}{2}\Psi^\dagger \sigma_2 \begin{pmatrix} 0 & iM \\ iM^T & 0 \end{pmatrix} \Psi^* - \frac{1}{2}\Psi^T \sigma_2 \begin{pmatrix} 0 & -iM^* \\ -iM^\dagger & 0 \end{pmatrix} \Psi. \quad (\text{B.35})$$

The contribution of the zero modes to the fermion determinant in the absence of diquark sources is given by [4, 88]

$$\det \begin{pmatrix} 0 & iM \\ iM^T & 0 \end{pmatrix}^{-\bar{\nu}} \quad \text{for } \bar{\nu} < 0 \quad \text{and} \quad \det \begin{pmatrix} 0 & -iM^* \\ -iM^\dagger & 0 \end{pmatrix}^{\bar{\nu}} \quad \text{for } \bar{\nu} \geq 0, \quad (\text{B.36})$$

where  $\bar{\nu} \equiv (n_R - n_L)/2$ .<sup>56</sup> Because of the degeneracy mentioned above the numbers  $n_R$ ,  $n_L$  of zero modes are even integers so that  $\bar{\nu} \in \mathbb{Z}$ . The number  $\bar{\nu}$  is proportional to  $\nu$ , the winding number of the gauge field. The proportionality constant depends on the color representation of the fermions. For example,  $\bar{\nu} = N_c \nu$  for fermions in the adjoint representation of  $\text{SU}(N_c)$  [4]. More general cases are considered, e.g., in [107].

The diquark source term can be written as

$$\frac{1}{2}\psi^T C(J_R P_R + J_L P_L)\psi + \text{h.c.} = -\frac{1}{2}\Psi^\dagger \sigma_2 \begin{pmatrix} J_R^\dagger & 0 \\ 0 & -J_L \end{pmatrix} \Psi^* - \frac{1}{2}\Psi^T \sigma_2 \begin{pmatrix} J_R & 0 \\ 0 & -J_L^\dagger \end{pmatrix} \Psi. \quad (\text{B.37})$$

Comparison with (B.35) again suggests a correspondence between diquark matrix and mass matrix, from which the contribution of the zero modes in the chiral limit and with nonzero diquark sources is given by

$$\det \begin{pmatrix} J_R & 0 \\ 0 & -J_L^\dagger \end{pmatrix}^{\bar{\nu}} = \det(-J_R J_L^\dagger)^{\bar{\nu}} \quad \text{for } \bar{\nu} \geq 0, \quad (\text{B.38a})$$

$$\det \begin{pmatrix} J_R^\dagger & 0 \\ 0 & -J_L \end{pmatrix}^{-\bar{\nu}} = \det(-J_R^\dagger J_L)^{-\bar{\nu}} \quad \text{for } \bar{\nu} < 0. \quad (\text{B.38b})$$

Collecting everything, we find for the partition function

$$Z(J_L, J_R) = \left\langle \left\{ \begin{array}{l} \det(-J_R J_L^\dagger)^{\bar{\nu}} \\ \det(-J_R^\dagger J_L)^{-\bar{\nu}} \end{array} \right\} \det''(D^\dagger D + J_R^\dagger J_R P_R + J_L^\dagger J_L P_L) \right\rangle_{\text{YM}} \quad \text{for } \begin{cases} \bar{\nu} \geq 0 \\ \bar{\nu} < 0 \end{cases}. \quad (\text{B.39})$$

Again, in deriving (B.39) we ignored accidental zero modes. The result can straightforwardly be extended to include them, and we then obtain (2.20).

<sup>55</sup>Again, the symbol  $\Psi$  used here should not be confused with the  $\Psi$  defined in (2.4) or after (B.20).

<sup>56</sup>Note that  $\bar{\nu}$  in [4, 88] is  $-\bar{\nu}$  in our notation, see footnote 52.

## C Consequences of a (non-) positive measure

### C.1 Diquark sources and positivity

From the results of appendix B we can immediately read off conditions for the positivity of the measure. Here we consider some cases of particular physical interest (assuming  $\theta = 0$ ).

- $\beta = 1$ : From (B.25) it follows that

1. The choice  $J_R = -J_L$  (source for the  $0^+$  diquark condensate) implies

$$\text{Pf}(J_R) \text{Pf}(J_L^\dagger) = \text{Pf}(J_R) \text{Pf}(-J_R^\dagger) = (-1)^{N_f} |\text{Pf}(J_R)|^2 = |\text{Pf}(J_R)|^2 > 0. \quad (\text{C.1})$$

Hence the measure is positive definite and there is no sign problem.

2. The choice  $J_R = J_L$  (source for the  $0^-$  diquark condensate) implies

$$\text{Pf}(J_R) \text{Pf}(J_L^\dagger) = \text{Pf}(J_R) \text{Pf}(J_R^\dagger) = (-1)^{N_f/2} |\text{Pf}(J_R)|^2. \quad (\text{C.2})$$

This is positive (negative) if  $N_f = 4n$  ( $N_f = 4n + 2$ ) with  $n \in \mathbb{N}$ . Thus the sign problem arises for  $N_f = 4n + 2$  due to the topological sectors with odd  $\nu$ .

- $\beta = 2$ : From (B.32) it follows that

1. The choice  $\rho = -\lambda$  (source for the  $0^+$  pion condensate) implies

$$-\rho\lambda^* = \rho\rho^* > 0 \quad (\text{C.3})$$

so that the measure is positive definite and there is no sign problem.

2. The choice  $\rho = \lambda$  (source for the  $0^-$  pion condensate) implies

$$-\rho\lambda^* = -\rho\rho^* < 0. \quad (\text{C.4})$$

Thus the sign problem arises due to the topological sectors with odd  $\nu$ .

- $\beta = 4$ : From (B.39) it follows that

1. The choice  $J_R = -J_L$  (source for the  $0^+$  diquark condensate) implies

$$\det(-J_R J_L^\dagger) = \det(J_R J_R^\dagger) > 0 \quad (\text{C.5})$$

so that the measure is positive definite and there is no sign problem.

2. The choice  $J_R = J_L$  (source for the  $0^-$  diquark condensate) implies

$$\det(-J_R J_L^\dagger) = (-1)^{N_f} \det(J_R J_R^\dagger). \quad (\text{C.6})$$

This is positive (negative) for even (odd)  $N_f$ ,<sup>57</sup> so the sign problem arises for odd  $N_f$  due to the topological sectors with odd  $\bar{\nu}$ .

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<sup>57</sup>Note that  $N_f$  is the number of Dirac fermions (not Majorana fermions).

## C.2 Sign problem and Dirac spectrum

### C.2.1 Discussion by Leutwyler and Smilga

A long time ago it was pointed out by Leutwyler and Smilga [4, Sec. VII] that the positivity of the measure is a necessary condition for the Banks-Casher relation [1] to hold. They used  $N_f = 1$  QCD at  $\mu = 0$  as an example. Here we review their discussion briefly and motivate our microscopic analysis in the next subsection.

A standard derivation of the Banks-Casher relation at  $\mu = 0$  goes as follows. Assuming  $\theta = 0$  for simplicity and taking  $m$  to be real (but not necessarily positive), we have

$$\begin{aligned}\langle \bar{\psi}\psi \rangle_{N_f=1} &= \frac{1}{V_4} \frac{\partial}{\partial m} \log \left\langle m^{|\nu|} \prod_n' (\lambda_n^2 + m^2) \right\rangle_{\text{YM}} \\ &= \frac{1}{V_4 m} \langle |\nu| \rangle_{N_f=1} + \frac{1}{V_4} \left\langle \sum_n' \frac{2m}{\lambda_n^2 + m^2} \right\rangle_{N_f=1} \\ &= \frac{1}{V_4 m} \langle |\nu| \rangle_{N_f=1} + \int_0^\infty d\lambda \rho(\lambda) \frac{2m}{\lambda^2 + m^2},\end{aligned}\tag{C.7}$$

where the  $i\lambda_n$  are the Dirac eigenvalues, the primed sum stands for the summation over nonzero modes, and

$$\rho(\lambda) = \frac{1}{V_4} \left\langle \sum_n' \delta(\lambda - \lambda_n) \right\rangle_{N_f=1}.\tag{C.8}$$

Note that  $\rho(\lambda)$  depends on  $m$  through the fermion determinant in the measure. The cases  $m > 0$  and  $m < 0$  differ qualitatively, as we shall see now.

- $m > 0$ : Since  $\langle \nu^2 \rangle_{N_f=1} = m V_4 \Sigma$  with  $\Sigma$  a low-energy constant, the first term in (C.7) is suppressed as  $O(1/\sqrt{V_4})$  and becomes negligible at large volume. The second term in (C.7) will reduce to  $\pi\rho(0)$  as  $m \rightarrow 0^+$  after the thermodynamic limit. Therefore

$$\langle \bar{\psi}\psi \rangle_{N_f=1} = \pi\rho(0).\tag{C.9}$$

- $m < 0$ : Since a probabilistic interpretation is no longer possible one cannot drop the first term of (C.7). Indeed we have [4]

$$\frac{1}{V_4 m} \langle |\nu| \rangle_{N_f=1} \sim \frac{\Sigma}{2\sqrt{2\pi}} \frac{e^{2|x|}}{|x|^{3/2}} \rightarrow \infty \quad \text{as } x = m V_4 \Sigma \rightarrow \infty.\tag{C.10}$$

Similarly  $\int_0^\infty d\lambda \rho(\lambda) \frac{2m}{\lambda^2 + m^2}$  diverges to  $-\infty$ , but these divergences cancel neatly in (C.7).

Now it is clear why the Banks-Casher relation (C.9) fails for  $m < 0$ :

- Zero-mode contributions are not suppressed at all in the thermodynamic limit. Non-trivial topologies cannot be ignored.
- The macroscopic spectral density  $\rho(\lambda)$  is ill-defined in the thermodynamic limit at least near the origin.

It seems natural to expect that these findings also apply to dense QCD-like theories, where the diquark (or pionic) sources can spoil the positivity of the measure (see section C.1). In a measure with indefinite sign,  $\rho_{\text{sv}}(\xi)$  would be singular in the thermodynamic limit, and we cannot derive Banks-Casher-type relations for the diquark and pion condensate. This is the reason why in the main text we chose only those sources that respect the positivity of the measure. An exceptional case is the limit  $\mu = \infty$ , where instantons are completely suppressed and the exact zero modes will disappear. For this case we can in principle choose any diquark/pionic sources.

We conclude this subsection with a few important remarks.

- Sometimes there are multiple external fields that couple to different condensates and can be inserted without spoiling the positivity. For instance, in  $N_f = 4$  QCD at  $\mu = 0$ , the Banks-Casher relation can be derived for both  $\langle \bar{\psi}_f \psi_f \rangle$  and  $\langle \bar{\psi}_f i\gamma_5 \psi_f \rangle$  because the degenerate *purely imaginary* mass term also respects positivity.<sup>58</sup> However,  $\rho(0)$  itself can be measured on the lattice without using any external sources. To which condensate  $\rho(0)$  is related to is determined by the infinitesimal external field we select.<sup>59</sup>
- The failure of a Banks-Casher relation does not prove the absence of the condensate<sup>60</sup> because it is possible that the condensate exists but is not determined by the value of  $\rho(\lambda)$  at the origin. An instructive example is three-color QCD at small  $\mu$ . It supports a nonzero  $\langle \bar{\psi} \psi \rangle$ , but there is no Banks-Casher relation. The microscopic spectral density of the complex Dirac eigenvalues exhibits a drastic oscillation with a period  $\sim O(1/V_4)$  and an amplitude  $\sim \exp(V_4)$  [108, 109]. This suggests that the *macroscopic* spectral density in the limit  $V_4 \rightarrow \infty$  is singular at least near the origin.

In the discussion by Leutwyler and Smilga, a sick behavior of  $\rho(\lambda)$  for  $N_f = 1$  with  $m < 0$  was alleged in an indirect way. In the next subsection we study the spectral density in the microscopic limit ( $\varepsilon$ -regime) and explicitly show how it behaves in an indefinite measure.

### C.2.2 The microscopic limit

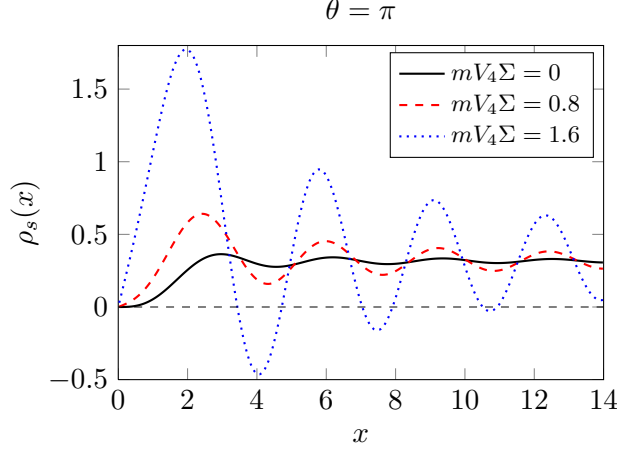
The effect of the topologically nontrivial sectors ( $\nu \neq 0$ ) on the Dirac spectrum was studied in detail in [110], where the microscopic spectral density and the chiral condensate in full QCD (including all topologies) were analyzed. All explicit examples in that reference were worked out for positive quark mass and vanishing CP-breaking angle  $\theta = 0$ . Here we demonstrate the effect of negative mass, or equivalently nonzero  $\theta$ , on the microscopic Dirac spectrum. As in [110], we use a formula [111] that expresses the microscopic correlation functions in terms of finite-volume partition functions. While in [110] the contributions from different topological sectors were simply summed up numerically, we here use a closed

<sup>58</sup>For  $N_f = 2$  this is not the case. This fact does not seem to have been appreciated in [50].

<sup>59</sup>More precisely, we have  $|\langle \bar{\psi} \psi \rangle| = |\langle \bar{\psi} i\gamma_5 \psi \rangle| = \pi\rho(0)$ , and the orientation of the condensate is determined by the external field. See also footnote 24.

<sup>60</sup>Indeed  $\langle \bar{q}q \rangle_{N_f=1}$  is the same for  $m > 0$  and  $m < 0$ .





**Figure 8.** Microscopic spectral density of the Dirac eigenvalues in  $N_c \geq 3$  QCD ( $\beta = 2$ ) for  $N_f = 1$  at  $\theta = \pi$  for different quark masses. Here  $\rho_s(x)$  is not necessarily positive definite and the probabilistic meaning as a “density” of eigenvalues is lost, reflecting the underlying sign problem.

expression for the full partition function summed over all topologies that became available later [112]. By doing this we avoid the errors coming from the truncation of an infinite to a finite sum.

Since all relevant formulas can be found in the literature [110–112] we omit the technical details and only show the final plot. As the simplest case we considered  $N_f = 1$  QCD at  $\mu = 0$  with  $N_c \geq 3$  ( $\beta = 2$ ). We have chosen  $\theta = \pi$  and  $m > 0$  so that the measure is not positive definite. (This is physically equivalent to  $\theta = 0$  with  $m < 0$ .) In figure 8 we show the plot of the microscopic spectral density with several values of the quark mass. In the chiral limit ( $m = 0$ ) the contribution from the  $\nu = 0$  sector is dominant and there is no effect of  $\theta$ . For nonzero quark mass, the microscopic spectral density exhibits a strong oscillation whose magnitude grows rapidly with  $mV_4\Sigma$ , while the period of oscillation looks roughly independent of  $mV_4\Sigma$ . This is exactly the opposite of what we observe at  $\theta = 0$ , where the dynamical quark decouples for a sufficiently large mass and  $\rho_s(x)$  becomes smoother. This oscillatory behavior and the failure of decoupling is reminiscent of the complex Dirac eigenvalue spectrum in QCD at  $\mu \neq 0$  [108]. We also checked that these characteristics are not limited to  $\theta = \pi$ : Even for a small  $\theta \neq 0$  the violent oscillation shows up eventually at a sufficiently large value of  $mV_4\Sigma$ .

Now we have understood microscopically why the standard Banks-Casher relation can fail: The spectral density varies rapidly over the scale of the quark mass and  $\rho(0)$  is certainly ill-defined in the thermodynamic limit. It is quite tempting to conjecture that this phenomenon is universal and will occur also in other theories with indefinite measure. In particular, we expect this for dense QCD-like theories with positivity-breaking external sources.<sup>61</sup> Indeed, in  $N_f = 2$  two-color QCD,  $j_R = j_L$  at  $\theta = 0$  is equivalent to  $j_R = -j_L$  at  $\theta = \pi$ , and the similarity to the case considered above is evident. It would be interesting

<sup>61</sup>The  $\theta$ -dependence of QCD-like theories at  $\mu \neq 0$  was studied in [113, 114], but the singular value spectrum of the Dirac operator was not considered there.

to study the impact of the sign problem on the singular value spectrum in more detail in analogy to the analysis of [108, 109].

### C.3 QCD inequalities

QCD inequalities are a powerful theoretical tool to impose strong constraints on the dynamics when the Euclidean functional measure is positive definite [115–118]. Although the sign problem hampers the application of QCD inequalities to three-color QCD at  $\mu \neq 0$ , they are still applicable to a class of dense QCD-like theories with positive definite measures [19, 21, 22, 26]. In particular it was shown for two-color QCD (with nonzero quark masses and no diquark sources) that the symmetry breaking is driven by the  $0^+$  diquark condensate [19], thus leaving parity unbroken. In this appendix we reexamine the application of QCD inequalities to two-color QCD at  $\mu \neq 0$  and extend the analysis to the case of nonzero diquark sources. For the sake of simplicity we will limit ourselves to either nonzero quark masses or nonzero diquark sources, but not both. We always work in Euclidean space.

#### C.3.1 Two-color QCD with quark masses

We review the original discussion in [19] and assume even  $N_f$  for positivity of the measure. In this subsection, the transpose ( $T$ ) and the adjoint ( $\dagger$ ) act only on color, spinor, and flavor indices, but not on space-time indices. The quark chemical potential  $\mu$  can be arbitrary in the following. When the diquark sources are absent and the (degenerate) quark mass is real and positive, the propagator in a fixed background gauge field is given by

$$S_{\psi\bar{\psi}}(x, y) \equiv \langle \psi(x) \bar{\psi}(y) \rangle_{\psi} = \left\langle x \left| \frac{1}{D + m} \right| y \right\rangle, \quad (\text{C.11})$$

where  $\langle \rangle_{\psi}$  denotes the average only over the fermion fields. Below we also use  $\langle \rangle_{\psi, A}$  and  $\langle \rangle_A$  for the full average and the average only over the gauge fields, respectively.

Let us consider a diquark operator  $M(x) \equiv \psi^T \Gamma \psi$  with an antisymmetric matrix  $\Gamma$  satisfying  $\Gamma^\dagger \Gamma = \mathbb{1}$ . The correlation function of this field is then given by

$$\begin{aligned} \langle M(x) M^\dagger(y) \rangle_{\psi, A} &= \langle \psi^T(x) \Gamma \psi(x) \bar{\psi}(y) \bar{\Gamma} \bar{\psi}^T(y) \rangle_{\psi, A} \quad (\bar{\Gamma} \equiv \gamma_4 \Gamma^\dagger \gamma_4) \\ &= \langle \text{tr} [\Gamma S_{\psi\bar{\psi}}(x, y) \bar{\Gamma} S_{\psi\bar{\psi}}(x, y)^T] \rangle_A \\ &\leq \langle \text{tr} [S_{\psi\bar{\psi}}(x, y) S_{\psi\bar{\psi}}(x, y)^\dagger] \rangle_A, \end{aligned} \quad (\text{C.12})$$

where the last line follows from the Cauchy-Schwarz inequality and the positivity of the measure. The same upper bound can be proven for those correlators of mesonic fields  $\bar{\psi} \Gamma \psi$  that have no disconnected piece.<sup>62</sup> On the other hand,  $(D + m)^T = C \tau_2 \gamma_5 (D^\dagger + m) C \tau_2 \gamma_5$

<sup>62</sup>By “disconnected piece” we mean a diagram whose quark lines are not connected directly, although they could be connected by gluons, see, e.g., figure 1 in [116]. The absence of disconnected pieces is essential for QCD inequalities. The correlators of  $\sigma$  and  $\eta'$  contain disconnected pieces and our arguments do not apply. Indeed, if the disconnected piece could be dropped, the  $\eta'$  would be as light as the pions. This is of course invalid. In reality the contribution of the gluonic intermediate state  $|F\tilde{F}\rangle$  is essential in the generation of the large mass of the  $\eta'$ .

allows us to write

$$\left\langle M(x)M^\dagger(y) \right\rangle_{\psi,A} = \left\langle \text{tr} [\Gamma S_{\psi\bar{\psi}}(x,y) \bar{\Gamma} C \tau_2 \gamma_5 S_{\psi\bar{\psi}}(x,y)^\dagger C \tau_2 \gamma_5] \right\rangle_A. \quad (\text{C.13})$$

If we assume  $\Gamma = C \tau_2 \gamma_5 \Gamma_f$  for an antisymmetric flavor matrix  $\Gamma_f$ , it follows that

$$\left\langle M(x)M^\dagger(y) \right\rangle_{\psi,A} = \text{tr}[\Gamma_f \Gamma_f^\dagger] \left\langle \text{tr} [S_{\psi\bar{\psi}}(x,y) S_{\psi\bar{\psi}}(x,y)^\dagger] \right\rangle_A. \quad (\text{C.14})$$

This is equal to the upper bound in (C.12) up to an irrelevant multiplicative constant,<sup>63</sup> implying that the diquarks in this channel are the lightest of all mesons and diquarks.<sup>64</sup> For  $\Gamma = C \tau_2 \Gamma_f$  the upper bound is not saturated, implying that if a diquark condensate forms, it does so in the  $0^+$  channel and not in the  $0^-$  channel. This is consistent with the observation that the instanton-induced interaction for  $N_f = 2$  is attractive in the  $0^+$  diquark channel and repulsive in the  $0^-$  diquark channel [69, 119, 120].

It is a general rule that the preferred direction of a condensate depends on the direction of the external symmetry-breaking field. In the case above, we considered real positive quark masses, but it is also instructive to consider other cases. Let us take degenerate purely imaginary masses for instance. For the sake of positivity ( $\det M > 0$ ) we require  $N_f = 4n$  with  $n \in \mathbb{N}$ . The propagator now reads

$$S_{\psi\bar{\psi}}(x,y) = \langle \psi(x) \bar{\psi}(y) \rangle_\psi = \left\langle x \left| \frac{1}{D + im\gamma_5} \right| y \right\rangle \quad \text{with } m \in \mathbb{R}, \quad (\text{C.15})$$

and because of  $(D + im\gamma_5)^T = -C \tau_2 (D + im\gamma_5)^\dagger C \tau_2$  the propagator satisfies  $S_{\psi\bar{\psi}}(x,y)^T = -C \tau_2 S_{\psi\bar{\psi}}(x,y)^\dagger C \tau_2$ . Thus we find from the second line of (C.12) that

$$\left\langle M(x)M^\dagger(y) \right\rangle_{\psi,A} = - \left\langle \text{tr} [\Gamma S_{\psi\bar{\psi}}(x,y) \bar{\Gamma} C \tau_2 S_{\psi\bar{\psi}}(x,y)^\dagger C \tau_2] \right\rangle_A. \quad (\text{C.16})$$

When  $\Gamma = C \tau_2 \Gamma_f$ , the r.h.s. reduces to  $\text{tr}[\Gamma_f \Gamma_f^\dagger] \left\langle \text{tr} [S_{\psi\bar{\psi}}(x,y) S_{\psi\bar{\psi}}(x,y)^\dagger] \right\rangle_A$ , saturating the inequality (C.12). Therefore this time the diquark condensation occurs in the  $0^-$  channel, breaking parity. The conclusion is that the quantum numbers of the condensate are not entirely determined by the internal dynamics of the system, but are sensitive to the external symmetry-breaking fields.

The reader may worry that this is in conflict with the Vafa-Witten theorem [121] stating that parity is not spontaneously broken in vector-like theories. Let us make two remarks on this point. First, various loopholes in the original “proof” have been discussed in the literature [122–128], and therefore it cannot be regarded as an established mathematical theorem. Second, our result does not contradict recent work [128] on a Vafa-Witten-type theorem for fermion bilinears, since the positivity of the probability distribution function

<sup>63</sup>The mass is given by the exponential decay of the correlation function, for which the multiplicative constant is irrelevant.

<sup>64</sup>Strictly speaking, this argument does not exclude the possibility that there are other mesons or diquarks of the same mass. For example, at  $\mu = 0$  there are mesonic states that have the same mass as the scalar diquarks due to the extended flavor symmetry. This degeneracy is lifted at  $\mu \neq 0$ .

of the observable, which plays an essential role in [128], is not ensured for purely imaginary masses.

We end with a brief remark on the rotation of the condensate. It is known from chiral perturbation theory that the chiral condensate at  $\mu = 0$  rotates into the diquark condensate for  $\mu > m_\pi/2$ . For real positive mass,  $\langle \bar{\psi}\psi \rangle$  at  $\mu = 0$  rotates into the  $0^+$  diquark condensate  $\langle \psi C \gamma_5 \psi \rangle$  [20]. For imaginary mass (and  $N_f = 4n$ ),  $\langle \bar{\psi} i \gamma_5 \psi \rangle$  at  $\mu = 0$  rotates into the  $0^-$  diquark condensate  $\langle \psi C \psi \rangle$ .<sup>65</sup>

### C.3.2 Two-color QCD with diquark sources

We now turn to two-color QCD in the chiral limit with diquark sources. In this subsection, the transpose and the adjoint again act only on color, spinor, and flavor indices, with one exception: as in the other parts of this paper the adjoint in  $D^\dagger$  acts on all indices.

In calculations of correlators, it is important to know whether there is a disconnected piece or not. If there is, the contribution of the gluonic intermediate states will invalidate the derivation of QCD inequalities. In the previous subsection, the annihilation of the diquark into gluons was trivially prohibited by the  $U(1)_B$  charge conservation, but the latter is explicitly violated once we insert diquark sources. Indeed, in the cases of  $J_R = \pm J_L = jI$  considered below, the correlators of  $\psi^T C \tau_2 I \psi$  and  $\psi^T C \tau_2 \gamma_5 I \psi$  have disconnected pieces, and QCD inequalities do not apply. We must limit ourselves to those correlators that have no disconnected piece.

The propagators in a fixed gauge field are given by the inverse of  $W$  defined in (2.6),

$$\begin{pmatrix} S_{\psi\psi}(x, y) & S_{\psi\bar{\psi}}(x, y) \\ S_{\bar{\psi}\psi}(x, y) & S_{\bar{\psi}\bar{\psi}}(x, y) \end{pmatrix} \equiv \begin{pmatrix} \langle \psi(x) \psi^T(y) \rangle_\psi & \langle \psi(x) \bar{\psi}(y) \rangle_\psi \\ \langle \bar{\psi}^T(x) \psi(y) \rangle_\psi & \langle \bar{\psi}^T(x) \bar{\psi}(y) \rangle_\psi \end{pmatrix} = \frac{1}{2} \langle x | W^{-1} | y \rangle. \quad (\text{C.17})$$

The operator inverse  $W^{-1}$  depends on the choice of the diquark sources. We first discuss the diquark sources with positive parity and then those with negative parity. As stated above we work in the chiral limit.

For the  $0^+$  diquark source, i.e.,  $J_R = -J_L = jI$  with real  $j$ , we find<sup>66</sup>

$$W^{-1} = \begin{pmatrix} -\frac{1}{j^2 + D^\dagger D} j C \tau_2 \gamma_5 I & \frac{1}{j^2 + D^\dagger D} D^\dagger \\ C \tau_2 D \frac{1}{j^2 + D^\dagger D} C \tau_2 & j C \tau_2 \gamma_5 I \frac{1}{j^2 + D D^\dagger} \end{pmatrix}. \quad (\text{C.18})$$

The positivity of the measure is ensured for even  $N_f$  (see section C.1). We now consider the correlator of the diquark field  $M(x) = \psi^T \Gamma \psi$ , assuming that there is no disconnected piece. A rerun of the arguments leading to (C.12) then shows that

$$\begin{aligned} \langle M(x) M^\dagger(y) \rangle_{\psi, A} &= \langle \psi^T(x) \Gamma \psi(x) \bar{\psi}(y) \bar{\Gamma} \bar{\psi}^T(y) \rangle_{\psi, A} \quad (\bar{\Gamma} \equiv \gamma_4 \Gamma^\dagger \gamma_4) \\ &= - \left\langle \text{tr} [\Gamma S_{\psi\bar{\psi}}(x, y) \bar{\Gamma} S_{\bar{\psi}\psi}(y, x)] \right\rangle_A \end{aligned}$$

<sup>65</sup>More general cases, not necessarily preserving the positivity of the measure, were addressed in [113] based on chiral perturbation theory with a nonzero  $\theta$ -angle.

<sup>66</sup>Note that the denominator in  $(W^{-1})_{22}$  contains  $DD^\dagger$  and not  $D^\dagger D$ . Using  $W$  in (2.6) one can explicitly confirm  $WW^{-1} = \mathbb{1}$ .

$$\leq \left\langle \text{tr} [S_{\psi\bar{\psi}}(x, y) S_{\psi\bar{\psi}}(x, y)^\dagger] \right\rangle_A. \quad (\text{C.19})$$

We now use (C.18) to show that<sup>67</sup>

$$\begin{aligned} S_{\psi\bar{\psi}}(x, y)^\dagger &= \left( \frac{1}{2} \left\langle x \left| \frac{1}{j^2 + D^\dagger D} D^\dagger \right| y \right\rangle \right)^\dagger \\ &= \frac{1}{2} \left\langle y \left| D \frac{1}{j^2 + D^\dagger D} \right| x \right\rangle \\ &= C\tau_2 S_{\bar{\psi}\psi}(y, x) C\tau_2. \end{aligned} \quad (\text{C.20})$$

Using this relation in the second line of (C.19) we obtain

$$\left\langle M(x) M^\dagger(y) \right\rangle_{\psi, A} = - \left\langle \text{tr} [\Gamma S_{\psi\bar{\psi}}(x, y) \bar{\Gamma} C\tau_2 S_{\psi\bar{\psi}}(x, y)^\dagger C\tau_2] \right\rangle_A. \quad (\text{C.21})$$

It is easily shown that for both  $\Gamma = C\tau_2\Gamma_f$  and  $\Gamma = C\tau_2\gamma_5\Gamma_f$ , we find

$$\left\langle M(x) M^\dagger(y) \right\rangle_{\psi, A} = \text{tr}[\Gamma_f \Gamma_f^\dagger] \left\langle \text{tr} [S_{\psi\bar{\psi}}(x, y) S_{\psi\bar{\psi}}(x, y)^\dagger] \right\rangle_A, \quad (\text{C.22})$$

which saturates the inequality (C.19) up to a multiplicative constant. Thus  $\psi^T C\tau_2\Gamma_f\psi$  and  $\psi^T C\tau_2\gamma_5\Gamma_f\psi$  are the lightest diquarks (with degenerate masses).

From  $S_{\psi\psi} \propto I$  we see that the nonexistence of disconnected pieces is ensured if  $\text{tr}[I\Gamma_f] = 0$ . Recalling from section 6.1 that an  $N_f \times N_f$  antisymmetric matrix can be expanded in the generators  $t^A$  of  $U(N_f)/\text{Sp}(N_f)$ , we find that this condition is satisfied for all  $A \neq 0$  but not for  $A = 0$ . Thus we conclude that  $\{\psi^T C\tau_2 t^A \psi\}_{A \neq 0}$  and  $\{\psi^T C\tau_2 \gamma_5 t^A \psi\}_{A \neq 0}$  are the lightest diquarks (with degenerate masses), whereas no information on  $\psi^T C\tau_2 I \psi$  and  $\psi^T C\tau_2 \gamma_5 I \psi$  can be gained from QCD inequalities.

The fields  $\{\psi^T C\tau_2 t^A \psi\}_{A \neq 0}$  and  $\{\psi^T C\tau_2 \gamma_5 t^A \psi\}_{A \neq 0}$  can be interpreted as the NG modes associated with the spontaneous symmetry breaking  $\text{SU}(N_f)_R \times \text{SU}(N_f)_L \rightarrow \text{Sp}(N_f)_R \times \text{Sp}(N_f)_L$ . However, this breaking is caused by *both*  $\langle \psi^T C\tau_2 I \psi \rangle$  and  $\langle \psi^T C\tau_2 \gamma_5 I \psi \rangle$ , which means that our result does not yield any constraint on the parity of the condensate. (This is to be contrasted with QCD with quark masses and no diquark sources, where we could determine the parity of the condensate even though we could not apply QCD inequalities to the correlators of  $\sigma$  and  $\eta'$ .)

We now turn to the  $0^-$  diquark source. For  $J_R = J_L = jI$  with real  $j$  we find

$$W^{-1} = \begin{pmatrix} -\frac{1}{j^2 + D^\dagger D} j C\tau_2 I & \frac{1}{j^2 + D^\dagger D} D^\dagger \\ C\tau_2 D \frac{1}{j^2 + D^\dagger D} C\tau_2 & -j C\tau_2 I \frac{1}{j^2 + D D^\dagger} \end{pmatrix}. \quad (\text{C.23})$$

For the positivity of the measure we require  $N_f = 4n$  (see section C.1). Since the off-diagonal entries are not changed from (C.18), our previous discussion goes through without modifications, and therefore we reach the same conclusion that  $\{\psi^T C\tau_2 t^A \psi\}_{A \neq 0}$  and

<sup>67</sup>Note that the interchange of  $x$  in  $y$  in the second line of (C.20) is correct even though the outer adjoint does not act on space-time indices.

$\{\psi^T C \tau_2 \gamma_5 t^A \psi\}_{A \neq 0}$  are the lightest diquarks. Again, the parity of the condensate cannot be determined from this argument alone.

A further discussion of the parity of the diquark condensate is given in the last paragraph of section 5.1.

## D Anomaly and index theorem for non-Hermitian Dirac operators

In this appendix we prove the following properties of the Euclidean Dirac operator<sup>68</sup>

$$D(\mu) \equiv \gamma_\nu D_\nu + \mu \gamma_4 \equiv \begin{pmatrix} 0 & D_L \\ D_R & 0 \end{pmatrix} \quad (\text{D.1})$$

at zero temperature and nonzero quark chemical potential  $\mu$  in a given gauge field:

1. The chiral anomaly equation in the chiral limit,

$$\partial_\nu J_{5\nu} = \frac{iN_f}{16\pi^2} F \tilde{F} \quad (\text{D.2})$$

with  $J_{5\nu} = i\bar{\psi} \gamma_\nu \gamma_5 \psi$ , is unchanged.

2. We have

$$\dim \ker D_R - \dim \ker D_R^\dagger = \nu, \quad (\text{D.3a})$$

$$\dim \ker D_L - \dim \ker D_L^\dagger = -\nu, \quad (\text{D.3b})$$

where  $\nu$  is the winding number of the gauge field, defined in (2.11).

3. For  $\mu = 0$ , (D.3) reduces to the ordinary index theorem<sup>69</sup>

$$\dim \ker D_R - \dim \ker D_L = \nu \quad (\text{D.4})$$

since  $D_R = -D_L^\dagger$  for  $\mu = 0$ .

The in-medium chiral anomaly and its impact on the phenomenology have been discussed in numerous studies [39, 54–57, 69, 104, 119, 120, 129–140]. So far the observation (D.2) has been made via perturbative calculations of the triangle diagram [130–132, 135, 138] as well as by Fujikawa’s path integral method [131, 135, 138]. The latter encounters some subtleties at  $\mu \neq 0$  that were not stressed in the preceding works. Also, in contrast to the chiral anomaly equation, the modification of the index theorem due to  $\mu \neq 0$  was rarely considered so far.

In this appendix we take a closer look at these issues. After some preliminaries in section D.1, we present in section D.2 two derivations of the anomaly equation (D.2) via

<sup>68</sup>The color generators are assumed to be in the fundamental representation of  $\text{SU}(N_c)$  for  $N_c \geq 2$ .

<sup>69</sup>Whether the r.h.s. of this equation is  $\nu$  or  $-\nu$  is determined by the convention for  $\gamma_5$ . Here we use  $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$  (see appendix A), which leads to  $+\nu$ . Some authors use the convention  $\gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4$ , which leads to  $-\nu$ . As mentioned in footnote 52, it seems that in [4] (although not stated explicitly) the second convention is used, which differs from ours by a sign.

the path integral method in a style that differs from [131, 135, 138]. In section D.3 we then study the index of the Dirac operator at  $\mu \neq 0$ . We will show that (D.3) can be proven for generic gauge fields and  $\mu \neq 0$ , while (D.4) requires an additional input. In section D.3.3 we will discuss where the difference between (D.3) and (D.4) comes from.

Three caveats are in order. First, we note that Fujikawa's analysis in its original form is equivalent to one-loop perturbation theory [141–144] although it is sometimes incorrectly said that it offers a nonperturbative derivation of the anomaly. Our analysis should be seen as the extension of this equivalence to  $\mu \neq 0$ . Second, our analysis does not extend to lattice fermions directly. It is beyond the scope of this paper to analyze the effects of  $\mu$  in the lattice formulation (see e.g., [145–147]). Third, the entire discussion in this appendix will be given for a fixed gauge field background.

## D.1 Preliminaries

### D.1.1 Path integral measure

To clarify our stance on the subject and fix the notation, we begin by reviewing Fujikawa's method at  $\mu = 0$  [148, 149]. It starts with the expansion of the fermion fields  $\psi$  and  $\bar{\psi}$  in terms of the eigenstates  $\{\psi_n\}$  of  $D(0)$ ,

$$\psi(x) = \sum_n a_n \psi_n(x) \equiv \sum_n a_n \langle x|n \rangle, \quad (\text{D.5a})$$

$$\bar{\psi}(x) = \sum_n \bar{b}_n \psi_n^\dagger(x) \equiv \sum_n \bar{b}_n \langle n|x \rangle, \quad (\text{D.5b})$$

where the  $a_n$  and  $\bar{b}_n$  are Grassmann variables. The transformations from  $\psi$  and  $\bar{\psi}$  to  $\{a_n\}$  and  $\{\bar{b}_n\}$  are unitary by virtue of the orthonormality and completeness of  $\{\psi_n\}$ , which are guaranteed since  $D(0)$  is anti-Hermitian. The functional measure then transforms as

$$\begin{aligned} \prod_x \mathcal{D}\psi(x) \mathcal{D}\bar{\psi}(x) &= (\det \langle n|x \rangle)^{-1} (\det \langle x|m \rangle)^{-1} \prod_n da_n d\bar{b}_n \\ &= (\det \langle n|m \rangle)^{-1} \prod_n da_n d\bar{b}_n = \prod_n da_n d\bar{b}_n, \end{aligned} \quad (\text{D.6})$$

where  $\det \langle n|x \rangle$  and  $\det \langle x|m \rangle$  each represent the determinant of an infinite-dimensional matrix specified by the indices  $n$ ,  $m$ , and  $x$ . By standard calculations the evaluation of the Jacobian for an infinitesimal chiral transformation leads to an infinite sum,  $\text{tr} \gamma_5 = \sum_n \psi_n(x)^\dagger \gamma_5 \psi_n(x)$ , which is to be regularized by, e.g.,  $e^{D^2/\Lambda^2}$  with  $\Lambda \rightarrow \infty$ .

Two remarks are in order. First, for the change of variables (D.6) to be unitary, the operator whose eigenstates are used to expand the fermion fields must be (anti-) Hermitian. Second, the final result does depend on the choice of the operator. Had we used the plane-wave basis, the result would simply have vanished:  $\text{tr}(\gamma_5 e^{\partial^2/\Lambda^2}) = 0$ . This is sometimes ascribed to the lack of gauge invariance, but the gauge invariance is merely a necessary condition. Indeed there is an instructive example,  $\text{tr}(\gamma_5 e^{D_\nu D_\nu/\Lambda^2}) = 0$ , which reveals that a seemingly unitary transformation from the gauge invariant eigenspace of  $D_\nu D_\nu$  to that of  $D$  is actually non-unitary in the presence of regularization [150].

The bottom line is that the operator used to define the functional space must be chosen so as to diagonalize the fermion action [78, 149]. The operators  $D_\nu D_\nu$  and  $\partial^2$  are not eligible, because their eigenbases do not diagonalize the action. In the following, we will work with this criterion as a guiding principle. This completes our preliminary comments on the chiral anomaly at  $\mu = 0$ .

### D.1.2 Remarks on the literature

$D(\mu)$  is no longer anti-Hermitian at  $\mu \neq 0$  and the eigenstates lose orthogonality, spoiling the unitary transformation (D.6). This requires us to choose a more appropriate definition of the path integral measure.

- In [131] the fermion fields were expanded in eigenstates of  $D(i\mu)$ , the Dirac operator with imaginary chemical potential. With  $D(i\mu)$  being Hermitian (in their convention), the eigenstates are orthonormal and the anomaly equation follows straightforwardly.
- In [135] the fermion fields were expanded in eigenstates of  $D(0)$ . Again the derivation of the anomaly equation is straightforward.
- In [138] the fermion fields were expanded in right and left eigenstates of  $D(\mu)$ , but the subtlety is that the “eigenstates” were simply defined by multiplying the eigenstates of  $D(0)$  by  $e^{\pm\mu x_4}$ . They either blow up exponentially at infinity for  $T = 0$ , or spoil the anti-periodic boundary condition in the  $x_4$  direction for  $T > 0$ . Note also that this multiplication does not change the eigenvalues from purely imaginary ones at  $\mu = 0$ . We suspect that the eigenstates thus obtained are not the legitimate eigenstates of  $D(\mu)$  (see, e.g., [56, 57]) since the contribution from occupied states below the Fermi surface is not taken into account [57].

The schemes based on  $D(i\mu)$  and  $D(0)$  [131, 135] do not meet our criterion, as the eigenstates of  $D(i\mu)$  and  $D(0)$  do not diagonalize the action. While the final result for the anomaly equation obtained in [131, 135, 138] is correct, the rigorousness of these approaches is not completely obvious to us.<sup>70</sup>

## D.2 Proofs of (D.2): Path integral derivation of the anomaly at $\mu \neq 0$

In this subsection we will present new derivations of the anomaly equation (D.2) by the path integral method, which we believe place the results in the literature on a firmer footing. For simplicity we assume  $N_f = 1$ . The extension to  $N_f > 1$  should be straightforward.

### D.2.1 Derivation based on $D^\dagger D$

A non-Hermitian Dirac operator occurs not only in QCD at  $\mu \neq 0$  but also in chiral gauge theories where the gauge interaction involves a  $\gamma_5$ -coupling. We shall apply one of the methods devised for this situation [151]<sup>71</sup> to QCD at  $\mu \neq 0$ . We use the same orthonormal

<sup>70</sup>Note also that the index theorem derived in [138] corresponds to (D.4), while the correct result at  $\mu \neq 0$  is (D.3) if there are accidental zero modes.

<sup>71</sup>In chiral gauge theories this regularization is known to give the so-called covariant anomaly [148].



bases  $\{\varphi_n\}$  and  $\{\tilde{\varphi}_n\}$  as in (B.2). Note again that the  $\xi_n$  are not the eigenvalues but rather the singular values of  $D$ , with  $\xi_n \geq 0$  for all  $n$ . The orthogonality and completeness of these bases follow from the Hermiticity of  $D^\dagger D$  and  $DD^\dagger$ . The fields can be expanded as

$$\psi(x) = \sum_n a_n \varphi_n(x), \quad \bar{\psi}(x) = \sum_n \bar{b}_n \tilde{\varphi}_n^\dagger(x), \quad (\text{D.7})$$

and the fermionic part of the action is then diagonalized,

$$S = \int d^4x \bar{\psi}(x) D \psi(x) = \sum_n \xi_n \bar{b}_n a_n. \quad (\text{D.8})$$

Therefore this scheme meets the criterion mentioned above. Note that, had we expanded both  $\psi$  and  $\bar{\psi}$  in only one of the bases, the action would not have been diagonalized.

Under an infinitesimal chiral transformation  $\psi \rightarrow e^{i\alpha\gamma_5} \psi$ ,  $\bar{\psi} \rightarrow \bar{\psi} e^{i\alpha\gamma_5}$ , the action changes as

$$e^{-S} \rightarrow e^{-S'} = \exp \left[ -S + \int d^4x \alpha(x) \partial_\nu J_{5\nu}(x) \right], \quad (\text{D.9})$$

while the functional measure also changes as

$$\begin{aligned} \mathcal{D}\psi \mathcal{D}\bar{\psi} &\rightarrow \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ -i \int d^4x \alpha(x) \sum_n \left[ \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) + \tilde{\varphi}_n^\dagger(x) \gamma_5 \tilde{\varphi}_n(x) \right] \right\} \\ &\equiv \lim_{\Lambda \rightarrow \infty} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ -i \int d^4x \alpha(x) \sum_n \left[ \varphi_n^\dagger(x) \gamma_5 e^{-D^\dagger D / \Lambda^2} \varphi_n(x) \right. \right. \\ &\quad \left. \left. + \tilde{\varphi}_n^\dagger(x) \gamma_5 e^{-DD^\dagger / \Lambda^2} \tilde{\varphi}_n(x) \right] \right\}, \end{aligned} \quad (\text{D.10})$$

where a Gaussian cutoff has been employed. Because the phase factor  $\alpha(x)$  is arbitrary, we have from (D.9) and (D.10)

$$\begin{aligned} \partial_\nu J_{5\nu} &= i \lim_{\Lambda \rightarrow \infty} \sum_n \left[ \varphi_n^\dagger(x) \gamma_5 e^{-D^\dagger D / \Lambda^2} \varphi_n(x) + \tilde{\varphi}_n^\dagger(x) \gamma_5 e^{-DD^\dagger / \Lambda^2} \tilde{\varphi}_n(x) \right] \\ &= i \frac{1}{32\pi^2} \varepsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}^a F_{\gamma\delta}^a, \end{aligned} \quad (\text{D.11})$$

where the last line is obtained using standard algebra involving the plane-wave basis. The  $\mu$ -dependent terms appear in the calculation but disappear from the final result. We have thus obtained the conventional anomaly equation (D.2).

### D.2.2 Derivation based on $D$

In the second derivation, we apply the method of [152, 153] to QCD at  $\mu \neq 0$ .<sup>72</sup> Let us denote the right and left eigenvectors of  $D(\mu)$  by  $\{\psi_n\}$  and  $\{\chi_n\}$ ,

$$D(\mu)\psi_n = \lambda_n \psi_n, \quad \chi_n^\dagger D(\mu) = \lambda_n \chi_n^\dagger. \quad (\text{D.12})$$

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<sup>72</sup>In chiral gauge theories this regularization is known to give the so-called consistent anomaly satisfying the Wess-Zumino consistency condition [148].

The eigenvalues  $\lambda_n$  are no longer purely imaginary but complex in general. We assume<sup>73</sup> that they satisfy the following orthonormality and completeness relations,<sup>74</sup>

$$\int d^4x \chi_m^\dagger(x) \psi_n(x) = \delta_{mn}, \quad \sum_n \psi_n(x) \chi_n^\dagger(y) = \delta^4(x-y). \quad (\text{D.13})$$

The orthonormalization is possible because the left and right eigenstates corresponding to different eigenvalues are orthogonal, as can be seen from

$$\lambda_m \int d^4x \chi_m^\dagger \psi_n = \int d^4x (\chi_m^\dagger D) \psi_n = \int d^4x \chi_m^\dagger (D \psi_n) = \lambda_n \int d^4x \chi_m^\dagger \psi_n. \quad (\text{D.14})$$

Let us expand the fields  $\psi$  and  $\bar{\psi}$  in the sets  $\{\psi_n\}$  and  $\{\chi_n^\dagger\}$ , respectively,

$$\psi(x) = \sum_n a_n \psi_n(x), \quad \bar{\psi}(x) = \sum_n \bar{b}_n \chi_n^\dagger(x), \quad (\text{D.15})$$

by which the fermion action is diagonalized as desired,

$$S = \int d^4x \bar{\psi} D \psi = \sum_k \sum_\ell \int d^4x \bar{b}_k \chi_k^\dagger(x) D a_\ell \psi_\ell(x) = \sum_k \lambda_k \bar{b}_k a_k. \quad (\text{D.16})$$

Thus this choice of bases also satisfies our criterion. After some algebra, the change of the measure under an infinitesimal chiral rotation is found to be

$$\begin{aligned} \mathcal{D}\psi \mathcal{D}\bar{\psi} &\rightarrow \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ -i \int d^4x 2\alpha(x) \sum_n \chi_n^\dagger(x) \gamma_5 \psi_n(x) \right] \\ &= \lim_{\Lambda \rightarrow \infty} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ -i \int d^4x 2\alpha(x) \sum_n \chi_n^\dagger(x) \gamma_5 \frac{1}{1 - D^2/\Lambda^2} \psi_n(x) \right]. \end{aligned} \quad (\text{D.17})$$

We avoided a Gaussian cutoff because complex eigenvalues are not suppressed by a Gaussian factor. Using the assumed completeness (D.13) we can move to the plane-wave basis. After some standard algebra, we find that the  $\mu$ -dependent terms disappear from the final result and recover the anomaly equation (D.2).

### D.3 Proofs of (D.3): Index theorem at $\mu \neq 0$

In the following we present two derivations of the index theorem at  $\mu \neq 0$ . The first derivation yields (D.3), while the second one yields (D.4). We analyze the origin of this discrepancy and relate it to the (in-) completeness of the bases.

#### D.3.1 Derivation based on $D^\dagger D$

Let us reexamine the analysis in section D.2.1. From (D.11) it follows that

$$\lim_{\Lambda \rightarrow \infty} \sum_n \int d^4x \left[ \varphi_n^\dagger(x) \gamma_5 e^{-D^\dagger D/\Lambda^2} \varphi_n(x) + \tilde{\varphi}_n^\dagger(x) \gamma_5 e^{-DD^\dagger/\Lambda^2} \tilde{\varphi}_n(x) \right] = 2\nu. \quad (\text{D.18})$$

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<sup>73</sup>This assumption can be violated on a gauge field set of measure zero, see section D.3.3, and then the derivation no longer works.

<sup>74</sup>The conditions in (D.13) say nothing about inner products such as  $\int d^4x \psi_m^\dagger(x) \psi_n(x)$ .

Using (B.3) one can show that the entire contribution to the l.h.s. comes solely from the eigenstates with  $\xi_n = 0$ , for any finite  $\Lambda$ . This observation, combined with

$$\sum_{n: \xi_n=0} \int d^4x \varphi_n^\dagger(x) \gamma_5 e^{-D^\dagger D/\Lambda^2} \varphi_n(x) = \dim \ker D_R - \dim \ker D_L \quad (\text{D.19})$$

and

$$\sum_{n: \xi_n=0} \int d^4x \tilde{\varphi}_n^\dagger(x) \gamma_5 e^{-DD^\dagger/\Lambda^2} \tilde{\varphi}_n(x) = \dim \ker D_L^\dagger - \dim \ker D_R^\dagger, \quad (\text{D.20})$$

proves the equality

$$\dim \ker D_R - \dim \ker D_L + \dim \ker D_L^\dagger - \dim \ker D_R^\dagger = 2\nu. \quad (\text{D.21})$$

On the other hand, using the identity

$$0 = \text{tr} \left( e^{-DD^\dagger/\Lambda^2} - e^{-D^\dagger D/\Lambda^2} \right) \quad (\text{D.22})$$

that follows from the cyclic invariance of the trace<sup>75</sup> and the fact that all nonzero eigenvalues of  $D^\dagger D$  and  $DD^\dagger$  coincide, we have

$$\dim \ker D = \dim \ker D^\dagger, \quad (\text{D.23})$$

or equivalently

$$\dim \ker D_R + \dim \ker D_L = \dim \ker D_R^\dagger + \dim \ker D_L^\dagger. \quad (\text{D.24})$$

Then (D.21) and (D.24) prove (D.3).

For  $\nu \geq 0$ ,  $\dim \ker D_R \geq \nu$  follows from (D.3a). For  $\nu < 0$ ,  $\dim \ker D_L \geq |\nu| = -\nu$  follows from (D.3b). Thus our index theorem at  $\mu \neq 0$  shows that  $D(\mu)$  must possess at least  $|\nu|$  zero modes for any  $\mu$ . This is consistent with the existence of exact zero modes in the instanton background at  $\mu \neq 0$  [55, 56].<sup>76</sup>

### D.3.2 Derivation based on $D$

Let us return to the second derivation of the anomaly in section D.2.2. From (D.17) and the anomaly equation (D.2) it follows that

$$\lim_{\Lambda \rightarrow \infty} \int d^4x \sum_n \chi_n^\dagger(x) \gamma_5 \frac{1}{1 - D^2/\Lambda^2} \psi_n(x) = \nu. \quad (\text{D.25})$$

If  $\lambda_n \neq 0$ , we have

$$\lambda_n \chi_n^\dagger(x) \gamma_5 \frac{1}{1 - \lambda_n^2/\Lambda^2} \psi_n(x) = [\chi_n^\dagger(x) D] \gamma_5 \frac{1}{1 - \lambda_n^2/\Lambda^2} \psi_n(x)$$

<sup>75</sup>It was pointed out in [154] that the cyclic invariance of the trace could potentially break down. This does not happen in the instanton background, see [154] or section 4.2 of [155]. This possibility does not invalidate our result (D.3) since we have also constructed an alternative proof of (D.3), using the eigenbases of  $D_5(\mu)$  and  $D_5(-\mu)$ , which does not make use of (D.22). For brevity we do not show this proof here.

<sup>76</sup>Note that the fermionic zero modes in the instanton background at  $\mu \neq 0$  have a norm that diverges logarithmically in the spatial volume [56].

$$\begin{aligned}
&= -\chi_n^\dagger(x) \gamma_5 \frac{1}{1 - \lambda_n^2/\Lambda^2} D\psi_n(x) \\
&= -\lambda_n \chi_n^\dagger(x) \gamma_5 \frac{1}{1 - \lambda_n^2/\Lambda^2} \psi_n(x), \tag{D.26}
\end{aligned}$$

and hence all contributions to the l.h.s. of (D.25) from nonzero modes vanish. Restricting the sum in (D.25) to zero modes, we arrive at

$$\dim \ker D_R - \dim \ker D_L \stackrel{?}{=} \nu. \tag{D.27}$$

This “proves” (D.4), i.e., the index theorem in the same form as at  $\mu = 0$ . It does not imply (D.3) proved in section D.3.1. The origin of this discrepancy is analyzed below.

### D.3.3 Analysis of the discrepancy

The arguments in section D.3.1 proved (D.3), whereas in section D.3.2 we were led to (D.4). Which is the true index theorem at  $\mu \neq 0$ ?

To answer this question, let us begin by recalling a standard argument at  $\mu = 0$  about the stability of the index of the Dirac operator against small perturbations. Suppose there are  $\nu$  right-handed zero modes and no left-handed zero modes in a given background gauge field. Now we deform the gauge field smoothly so that two of the eigenvalues  $\lambda$  and  $-\lambda$  (with eigenstates  $\psi$  and  $\gamma_5\psi$ , respectively) approach zero. When  $\lambda = 0$  is achieved, we have two accidental zero modes, which we can arrange to be eigenstates of  $\gamma_5$  by choosing the linear combinations  $(\mathbb{1} + \gamma_5)\psi$  and  $(\mathbb{1} - \gamma_5)\psi$ . Thus the number of right-handed (left-handed) zero modes becomes  $\nu + 1$  (1), but their difference never changes:  $(\nu + 1) - 1 = \nu$ . This is why accidental zero modes cannot change the index.

This argument is correct at  $\mu = 0$ . If it could be extended to  $\mu \neq 0$ , the accidental zero modes could not change  $D = \dim \ker D_R - \dim \ker D_L$  and the ordinary index theorem (D.4) would continue to hold. However, the argument does not work for  $\mu \neq 0$ . The pitfall is that, when  $D(\mu)$  is not anti-Hermitian,  $\psi$  and  $\gamma_5\psi$  may become linearly dependent in the limit  $\lambda \rightarrow 0$ . Let us exemplify this by a toy model which mimics  $D(\mu)$ :

$$D_{\text{toy}}(\alpha) = \begin{pmatrix} 0 & 0 & 2 & -4 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix} \quad \text{with eigenvalues} \quad \{\pm 2, \pm \sqrt{2\alpha}\}. \tag{D.28}$$

The right eigenvectors of  $D_{\text{toy}}(\alpha)$  associated with the eigenvalues  $\sqrt{2\alpha}$  and  $-\sqrt{2\alpha}$  are  $(4, -2, 0, -\sqrt{2\alpha})^T$  and  $(4, -2, 0, \sqrt{2\alpha})^T$ , respectively. In the limit  $\alpha \rightarrow 0$  they coincide, implying that  $\dim \ker D_R$  increases by 1 whereas  $\dim \ker D_L$  remains zero. The accidental zero modes changed the index.

At  $\alpha = 0$  the dimension of the eigenspace of  $D_{\text{toy}}(\alpha)$  is 3 ( $< 4$ ). This means that the eigenvectors fail to form a complete basis, a phenomenon that occurs only for non-Hermitian matrices. The interpretation of the discrepancy between the last two subsections is now straightforward: Our derivation in section D.3.2 led to (D.4) because we assumed the completeness of the basis in (D.13). If the completeness is not ensured, it is not possible

to expand an arbitrary fermion field in this basis as in (D.7), and then our discussions in section D.2.2 and D.3.2 break down.<sup>77</sup> The index theorem that holds for generic gauge fields and  $\mu \neq 0$  is not (D.4), but (D.3).

However, the incompleteness of the basis requires a fine-tuning of the gauge field, and so we expect that this occurs only on a gauge field set of measure zero. For practical calculations it is justified to neglect the possibility of incompleteness, in the same sense as we can neglect accidental (non-topological) zero modes at  $\mu = 0$ . Thus both (D.3) and (D.4) will be valid for almost all gauge fields and  $\mu \neq 0$ . Summarizing, we have

$$\text{for } \nu \geq 0 : \dim \ker D_R \stackrel{\text{a.s.}}{=} \dim \ker D_L^\dagger \stackrel{\text{a.s.}}{=} \nu, \quad \dim \ker D_R^\dagger \stackrel{\text{a.s.}}{=} \dim \ker D_L \stackrel{\text{a.s.}}{=} 0, \quad (\text{D.29a})$$

$$\text{for } \nu < 0 : \dim \ker D_R^\dagger \stackrel{\text{a.s.}}{=} \dim \ker D_L \stackrel{\text{a.s.}}{=} -\nu, \quad \dim \ker D_R \stackrel{\text{a.s.}}{=} \dim \ker D_L^\dagger \stackrel{\text{a.s.}}{=} 0, \quad (\text{D.29b})$$

where  $\stackrel{\text{a.s.}}{=}$  denotes an equality that holds “almost surely”. This completes our discussion of the index theorem at  $\mu \neq 0$ .

## E Derivation of (6.19)

In this appendix we outline the derivation of the singular value density in the free limit in (6.19). The basic relation we use is

$$\rho_{\text{sv}}(\xi) = -\frac{2\xi}{\pi} \text{Im} \left\langle \sum_n \frac{1}{\xi_n^2 - \xi^2 + i\varepsilon} \right\rangle, \quad (\text{E.1})$$

where  $\varepsilon \rightarrow 0^+$  is tacitly assumed. To obtain the resolvent on the r.h.s., we express it in terms of the free Dirac operator  $D(\mu) = (\gamma_\nu \partial_\nu + \mu \gamma_4) \otimes \mathbb{1}_{N_c}$  ( $\mu > 0$ ) and move to momentum space,

$$\left\langle \sum_n \frac{1}{\xi_n^2 - \xi^2 + i\varepsilon} \right\rangle = \left\langle \text{tr} \frac{1}{D(\mu)^\dagger D(\mu) - \xi^2 + i\varepsilon} \right\rangle \quad (\text{E.2})$$

$$= V_4 N_c \int \frac{d^4 p}{(2\pi)^4} \text{tr} \frac{1}{(-i\not{p} + \mu\gamma_4)(i\not{p} + \mu\gamma_4) - \xi^2 + i\varepsilon}. \quad (\text{E.3})$$

Then

$$\text{Im} \left\langle \sum_n \frac{1}{\xi_n^2 - \xi^2 + i\varepsilon} \right\rangle = -4\pi V_4 N_c \int \frac{d^4 p}{(2\pi)^4} |p^2 + \mu^2 - \xi^2| \delta((p^2 + \mu^2 - \xi^2)^2 - 4\mu^2 \mathbf{p}^2), \quad (\text{E.4})$$

where  $\mathbf{p} = (p_1, p_2, p_3)$  denotes the three-momentum. After elementary but tedious algebra this integral can be performed analytically and (6.19) follows.

<sup>77</sup>Although our discussion here is based on finite-dimensional matrices, we believe that the essential part of the argument carries over to the actual Dirac operator as an infinite-dimensional matrix.

## F Random matrix theory for QCD with isospin chemical potential ( $\beta = 2$ )

In this appendix, we comment on the random matrix theory describing QCD with three colors, two flavors, and isospin chemical potential  $\mu_I = 2\mu$ , see sections 2.2 and 4.2. Starting from the two-matrix model of [156] the random matrix theory for this case in the chiral limit is

$$Z_{\nu, N_f=2}^{\text{iso}}(\hat{\mu}, \hat{\rho}_1, \hat{\rho}_2) = \int dA dB e^{-N \text{tr}(A^\dagger A + B^\dagger B)} \det \begin{pmatrix} D(\hat{\mu}) & X \\ Y & D(-\hat{\mu}) \end{pmatrix} \quad (\text{F.1})$$

with

$$D(\hat{\mu}) = \begin{pmatrix} 0 & iA + \hat{\mu}B \\ iA^\dagger + \hat{\mu}B^\dagger & 0 \end{pmatrix}, \quad (\text{F.2})$$

where  $A$  and  $B$  are complex  $N \times (N + \nu)$  matrices and

$$X = \begin{pmatrix} \hat{\rho}_1 \mathbb{1}_N & 0 \\ 0 & \hat{\rho}_2 \mathbb{1}_{N+\nu} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \hat{\rho}_2 \mathbb{1}_N & 0 \\ 0 & \hat{\rho}_1 \mathbb{1}_{N+\nu} \end{pmatrix} \quad (\text{F.3})$$

are source terms for the pion condensate. This model can be extended to  $N_f$  pairs of quarks and conjugate quarks. Introducing  $P = iA + \hat{\mu}B$  and  $Q = iA^\dagger + \hat{\mu}B^\dagger$ , we obtain

$$Z_{\nu, N_f=2}^{\text{iso}}(\hat{\mu}, \hat{\rho}_1, \hat{\rho}_2) = \int dP dQ \exp \left\{ -\frac{N(1 + \hat{\mu}^2)}{4\hat{\mu}^2} \left[ \text{tr}(P^\dagger P + Q^\dagger Q) + \text{tr}(PQ + Q^\dagger P^\dagger) \right] \right\} \\ \times \hat{\rho}_1^{-\nu} \hat{\rho}_2^\nu \det(P^\dagger P + \hat{\rho}_1^2 \mathbb{1}_{N+\nu}) \det(Q^\dagger Q + \hat{\rho}_2^2 \mathbb{1}_N). \quad (\text{F.4})$$

To study the singular values of  $D(\hat{\mu})$  we need to express the partition function in terms of the eigenvalues of  $P^\dagger P$  and  $Q^\dagger Q$ . Now, we notice that (F.4) is identical to eqs. (2.5)–(2.8) of [157].<sup>78</sup> The representation of (F.4) in terms of the eigenvalues of  $P^\dagger P$  and  $Q^\dagger Q$  is given in eq. (2.14) of that reference, and the microscopic spectral correlations were also computed in [157]. Hence, the microscopic correlations of the singular values of  $D(\hat{\mu})$  can be obtained immediately from the results of [157]. The mapping of RMT parameters to physical quantities can be worked out similarly to the  $\beta = 1$  case in section 7.

## G Derivation of (7.52)

From (7.51) we have for  $\nu = 0$

$$\left\langle \sum_n' \frac{1}{\xi_n^2} \right\rangle_0 = 2(V_4 \Phi_{\mathbf{L}})^2 \frac{\partial}{\partial z} \ln f(z) \quad \text{with} \quad f(z) = \int_{S^5} d\vec{n} e^{z(n_2^2 + n_4^2)}. \quad (\text{G.1})$$

Performing a Hubbard-Stratonovich transformation gives

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp dq e^{-p^2 + q^2} \int_{S^5} d\vec{n} e^{2\sqrt{z}(pn_2 + qn_4)}. \quad (\text{G.2})$$

<sup>78</sup>It is interesting that the model of [157] is useful here since it applies to QCD with imaginary chemical potential, which is unrelated to QCD with real isospin chemical potential.

We now define  $\vec{a} = (0, 2\sqrt{z}p, 0, 2\sqrt{z}q, 0, 0)$  and then rotate the coordinate system such that  $\vec{a}$  is parallel to  $\hat{n}_1$  to obtain

$$\begin{aligned} \int_{S^5} d\vec{n} e^{2\sqrt{z}(pn_2+qn_4)} &= \int_{S^5} d\vec{n} e^{a n_1} \quad \text{with} \quad a = |\vec{a}| \\ &= \frac{8\pi^2}{3} \int_0^\pi d\varphi \sin^4\varphi e^{a \cos\varphi} = \frac{8\pi^3}{a^2} I_2(a). \end{aligned} \quad (\text{G.3})$$

After introducing polar coordinates for  $p$  and  $q$  we end up with

$$f(z) = \frac{4\pi^3}{z} \int_0^\infty dR \frac{e^{-R^2}}{R} I_2(2\sqrt{z}R) = \pi^3 M(1, 3, z), \quad (\text{G.4})$$

where  $M$  is Kummer's function (a.k.a. the confluent hypergeometric function). Using the recurrence relations for  $M$  we obtain

$$M(1, 3, z) = \frac{2}{z} \left( \frac{e^z - 1}{z} - 1 \right), \quad (\text{G.5})$$

and after performing the differentiation according to (G.1) we obtain (7.52).

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